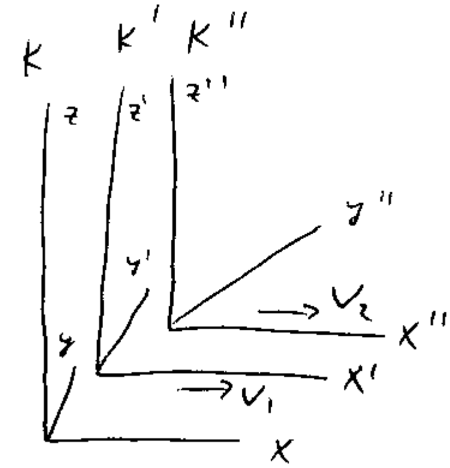
To determine f consider 3 inertial frames K, K' & K'':



$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & -v_1 \\ -v_1 f_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \alpha_2 \begin{pmatrix} 1 & -v_2 \\ -v_2 f_2 & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix} =$$

$$= \alpha_1 \alpha_2 \begin{pmatrix} 1 & -v_2 \\ -v_2 f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -v_1 \\ -v_1 f_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \alpha_1 \alpha_2 \cdot$$

$$\cdot \begin{pmatrix} 1 + v_1 v_2 f_1 & -v_1 - v_2 \\ -v_2 f_2 - v_1 f_1 & 1 + v_1 v_2 f_2 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$\Rightarrow$ should be also equal to $\alpha(u) \begin{pmatrix} 1 & -u \\ -u f(u) & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$

$\Rightarrow f_1 = f_2 \Rightarrow f(v_1) = f(v_2)$ ~ a constant!

Denote $f = \frac{1}{c^2}$.

1) $c = \infty \Rightarrow f = 0 \Rightarrow$ get Galilean transformations

2) c ~ finite $\Rightarrow \frac{1}{\sqrt{1 - v^2/c^2}}$ ~ real $\Rightarrow$ any $v < c$

$\Rightarrow$ maximum limiting speed!

We get Lorentz transformations

$$\boxed{x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y' = y, \ z' = z}$$

The velocity transforms as: (check the double transform above)

$$u = \frac{v_1 + v_2}{1 + v_1 v_2 f} \Rightarrow u = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

(if $v_1 = c$ & $v_2 = c \Rightarrow u = c$!)

Define 4-dim coordinates: $x_0 = ct$, $x_1 = x$
$x_2 = y$, $x_3 = z$

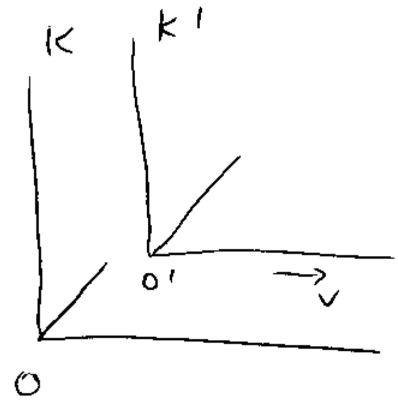
$\Rightarrow$ also Define $\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$, $\beta \equiv \frac{v}{c}$

$$\Rightarrow \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The inverse Lorentz transform:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

Invariant interval: flash a light at the origin(s) at $t = t' = 0$



$\Rightarrow$ light reaches point (x, y, z) after time t such that

$$c^2t^2 - x^2 - y^2 - z^2 = 0$$

In moving frame (K') have

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = 0.$$

$\Rightarrow$ as $y' = y$, $z' = z$ $\Rightarrow$ $c^2t^2 - x^2 = c^2t'^2 - x'^2$

$\Rightarrow$ can explicitly check that it works:

$$c^2t'^2 - x'^2 = \frac{1}{1 - \frac{v^2}{c^2}} \left[\left(ct - \frac{v}{c}x\right)^2 - (x - vt)^2 \right] =$$

$$= \frac{1}{1 - \frac{v^2}{c^2}} \left[c^2t^2\left(1 - \frac{v^2}{c^2}\right) - x^2\left(1 - \frac{v^2}{c^2}\right) \right] = c^2t^2 - x^2$$

as expected.

Quantity $s_{12}^2 = c^2 (t_1^2 - t_2^2) - |\vec{x}_1 - \vec{x}_2|^2$

is called the square of the interval between 2 events

at times t_1 & t_2 & locations $\vec{x}_1$ & $\vec{x}_2$.

(i) $s_{12}^2 > 0$ => timelike separation => there exists a frame where $\vec{x}_1' = \vec{x}_2'$ => $s_{12}^2 = c^2(t_1'^2 - t_2'^2)$

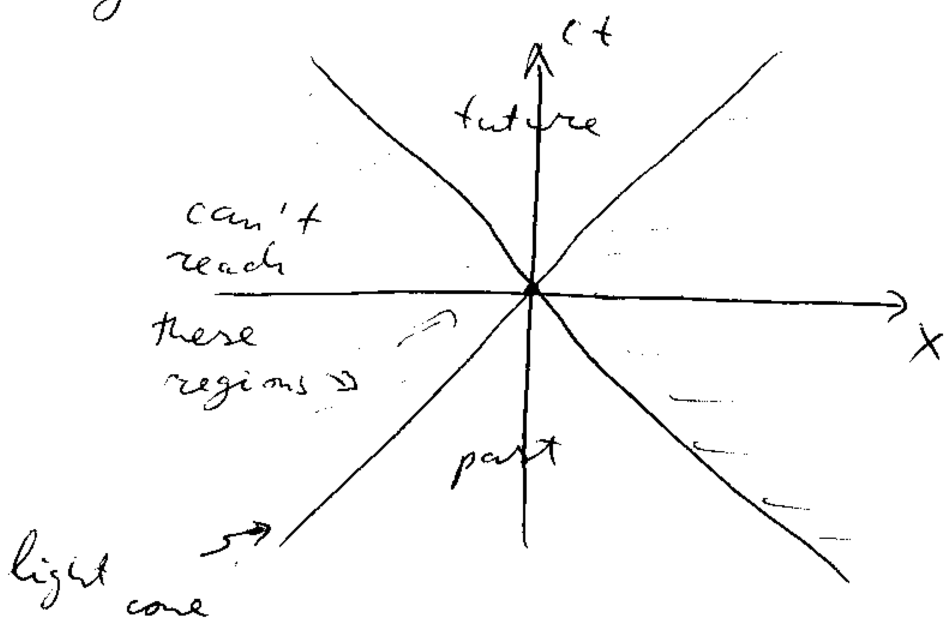
=> the events take place at the same space point, but at diff. times

(ii) $s_{12}^2 < 0$ => spacelike separation =>

there exists a frame where $t_1'' = t_2''$ =>

=> $s_{12}^2 = -(\vec{x}_1'' - \vec{x}_2'')^2$ => events take place at the same time but at different locations

(iii) $s_{12}^2 = 0$ => lightlike separation



Proper time & Time Dilation

Definition Proper time is the time in the rest frame of an object: $\boxed{d\tau \equiv \frac{ds}{c}}$.

Example: imagine a frame in which a particle moves with velocity $\vec{u}(t) \Rightarrow d\vec{x} = \vec{u}(t)\,dt$

$$\Rightarrow ds^2 = c^2 dt^2 - (d\vec{x})^2 = c^2 dt^2 - \vec{u}^2 dt^2 =$$

$$= c^2 dt^2 \left(1 - \beta^2(t)\right).$$

In the ~~sy~~ rest frame of the particle

$$d\tau = \frac{ds}{c} = dt\sqrt{1-\beta^2(t)} \Rightarrow \boxed{\tau_2 - \tau_1 = \int_{t_1}^{t_2} dt\sqrt{1-\beta^2(t)}}.$$

alternatively

$$\boxed{t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1-\beta^2(\tau)}}} \Rightarrow \Delta t \geq \Delta\tau$$

~ time dilation.

(E.g.: a photon has $\beta = 1 \Rightarrow \Delta\tau = 0 \Rightarrow$

$\Rightarrow$ the whole lifetime of the Universe is instanteneous for a photon!)