which gives

$$E_x(\omega) = -\frac{2iq\omega}{V^2(2\pi)^{3/2}}\left[\frac{1}{\varepsilon(\omega)}-\beta^2\right]\int_{-\infty}^{\infty} dk_y\, dk_z\, e^{ik_y b}$$

$$\cdot \frac{1}{k_y^2+k_z^2+\lambda^2}$$

The k_z-integral is $\displaystyle\int_{-\infty}^{\infty}\frac{dk_z}{k_z^2+k_y^2+\lambda^2} = 2\pi i\,\frac{1}{2i\sqrt{k_y^2+\lambda^2}} =$

$$= \frac{\pi}{\sqrt{k_y^2+\lambda^2}} \Rightarrow$$

$$E_x(\omega) = -\frac{iq\omega}{V^2(2\pi)^{1/2}}\left[\frac{1}{\varepsilon(\omega)}-\beta^2\right]\underbrace{\int_{-\infty}^{\infty} dk_y\, e^{ik_y b}\frac{1}{\sqrt{k_y^2+\lambda^2}}}_{2K_0(\lambda b)\ (\mathrm{Re}\,\lambda^2>0)}$$

$$\Rightarrow E_x(\omega) = -\frac{iq\omega}{V^2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{\varepsilon(\omega)}-\beta^2\right]K_0(\lambda b)$$

Other non-zero components are

$$E_y(\omega) = \frac{q}{V}\sqrt{\frac{2}{\pi}}\,\frac{\lambda}{\varepsilon(\omega)}\,K_1(\lambda b)$$

$$B_z(\omega) = \varepsilon(\omega)\,\beta\, E_y(\omega)$$

$\Rightarrow$ energy loss on a single electron is

$$\Delta \mathcal{E}(b) = 2|e| \, \mathrm{Re} \int_0^\infty d\omega \; i\omega \, \vec{x}(\omega) \cdot \vec{E}^*(\omega)$$

Note that $-|e| \, \vec{x}(\omega)$ is the dipole moment of an electron $\Rightarrow$ if n is the number density of electrons in a volume $\Rightarrow$ a volume ~~[illegible]~~ element d^3x has $n \, d^3x$ electrons $\Rightarrow$ their combined dipole moment is $-|e| \, n \, \vec{x}(\omega) \, d^3x = \vec{P} \, d^3x$ as $\vec{P}$ (polarization) is the electric dipole moment per unit volume $\Rightarrow$ replace $-|e| \, \vec{x} \to -|e| \, n \, \vec{x} = \vec{P}$

Finally, as $\vec{P}(\omega) = \frac{\varepsilon(\omega) - 1}{4\pi} \vec{E}(\omega)$ (Gaussian units)

$\Rightarrow$ the energy loss is

$$\Delta \mathcal{E}(b) = -\frac{1}{2\pi} \, \mathrm{Re} \int_0^\infty d\omega \; i\omega \left[\varepsilon(\omega) - 1\right] \vec{E}(\omega) \cdot \vec{E}^*(\omega)$$

Now it has the meaning of energy loss per unit volume $\Rightarrow$ makes sense to integrate over impact parameter d^2b to obtain energy loss per unit length:

$$\left(\frac{d\mathcal{E}}{dx}\right)_{b>a} = \int_a^\infty db \cdot 2\pi b \; \Delta\mathcal{E} = \int_a^\infty db \cdot b \cdot \mathrm{Re}\left\{\int_0^\infty d\omega \, (-i\omega) \cdot\right.$$

$$\left. \cdot \left[\varepsilon(\omega) - 1\right] \left|\vec{E}(\omega)\right|^2 \right\},$$ where a is some minimal impact parameter, ~ size of an atom.

Now, as $|\vec{E}(\omega)|^2 = |E_x(\omega)|^2 + |E_y(\omega)|^2 =$

$$= \frac{q^2\omega^2}{V^4} \frac{2}{\pi} \left[\frac{1}{\varepsilon} - \beta^2\right]^2 K_0(\lambda b) \cdot K_0(\lambda^* b) + \frac{q^2}{V^2} \frac{2}{\pi} \frac{\lambda^2}{\varepsilon^2} \cdot$$

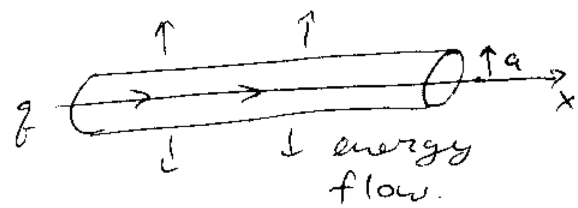
$\cdot K_1(\lambda b) K_1(\lambda^* b) \Rightarrow$ after some algebra and calculus get

$$\boxed{\left(\frac{d\mathcal{E}}{dx}\right)_{b>a} = \frac{2}{\pi} \frac{q^2}{V^2} \mathrm{Re} \int_0^\infty d\omega \; i\omega \lambda^* a \, K_1(\lambda^* a) K_0(\lambda a) \cdot \left[\frac{1}{\varepsilon(\omega)} - \beta^2\right]}$$

$\lambda^2 = \frac{\omega^2}{V^2}\left[1 - \beta^2 \varepsilon\right]$

E. Fermi, ~1940

Alternatively, surround particle's trajectory by a cylinder of radius a $\Rightarrow$



$$\left(\frac{d\mathcal{E}}{dx}\right)_{b>a} = \frac{1}{V}\frac{d\mathcal{E}}{dt} = -\frac{c}{4\pi V} \int_{-\infty}^{\infty} d\cancel{x} \; 2\pi a \cdot B_z E_x \left(= +\frac{1}{V} \int_{-\infty}^{\infty} d\cancel{x} \cdot 2\pi a \cdot S_n \right)$$

Poynting vector

$$= -\frac{c}{4\pi} \cdot 2\pi a \int_{-\infty}^{\infty} dt \; B_z E_x = -ca \, \mathrm{Re} \int_0^\infty d\omega \; B_z^*(\omega) E_x(\omega)$$

& plug in the fields.

Non-relativistic case: $\beta \ll 1 \Rightarrow \lambda = \frac{\omega}{v} \sim$ real

$\Rightarrow K_1(\lambda^* a) K_0(\lambda a) \sim$ real $\Rightarrow$ only $\frac{1}{\varepsilon(\omega)}$ may have

Im part $\Rightarrow$ $$\left(\frac{d\mathcal{E}}{dx}\right)_{b>a} \approx -\frac{2}{\pi} \frac{q^2}{v^2} \int_0^\infty d\omega \cdot \omega \cdot \lambda a K_1(\lambda a) K_0(\lambda a) \cdot \operatorname{Im} \frac{1}{\varepsilon(\omega)}$$

$\Rightarrow$ energy loss probes $\operatorname{Im} 1/\varepsilon(\omega)$ $\Rightarrow$ $\varepsilon(\omega)$ has to have Im part for energy loss to take place.

$$\left(\text{e.g. } \varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}.\right)$$

Ultrarelativistic case $\beta \simeq 1 \Rightarrow \lambda \simeq \frac{\omega}{v}\sqrt{1-\varepsilon(\omega)}$

Assume that $\lambda a \ll 1$ $\Rightarrow K_1(z) \simeq \frac{1}{z}$, $z \ll 1$

$K_0(z) \simeq -\ln z + \ln 2 - \gamma$, $z \ll 1$

$$\Rightarrow \left(\frac{d\mathcal{E}}{dx}\right)_{b>a} = \frac{2}{\pi}\frac{q^2}{c^2} \operatorname{Re} \int_0^\infty d\omega \cdot i\omega \cdot \ln\left(\frac{1.123}{\lambda a}\right)\left[\frac{1}{\varepsilon} - 1\right] =$$

$$= \frac{2}{\pi}\frac{q^2}{c^2} \operatorname{Re} \int_0^\infty d\omega \cdot i\omega \left[\frac{1}{\varepsilon(\omega)} - 1\right] \cdot \left\{\ln\left(\frac{1.123c}{\omega a}\right) - \frac{1}{2}\ln(1-\varepsilon)\right\}.$$

Try $\varepsilon(\omega) = 1 + \dfrac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}$ $\Rightarrow$ after some algebra get

$$\boxed{\left(\frac{d\mathcal{E}}{dx}\right)_{b>a} = \frac{q^2\omega_p^2}{c^2}\ln\left(\frac{c\cdot 1.123}{a\omega_p}\right)}$$

$\Rightarrow$ always get a log of the ratio of frequencies

the highest allowed frequency is c/a

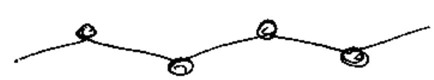
the lowest — " — is ω_p ~ cut off by medium effects.

$\Rightarrow \omega_p^2 \propto \rho$ ~ material's density

$\Rightarrow$ people talk of energy loss in $\frac{\text{MeV}}{\text{cm}}\frac{1}{\rho} =$

$= \frac{\text{MeV}}{\text{cm}}\frac{\text{cm}^3}{\text{g}} = \text{MeV}\frac{\text{cm}^2}{\text{g}}$

$\Rightarrow$ There is also collisional (mechanical) energy loss



$\Rightarrow$ Energy loss typically is of order of $1-2\ \text{MeV}\frac{\text{cm}^2}{\text{g}}$ for most media.

(E.g. cosmic rays in the atmosphere).

Cherenkov Radiation

Imagine a particle in the medium with $\varepsilon(\omega)$ moving with the velocity > than the phase velocity

$$v_{ph} = \frac{\omega}{k} = \frac{\omega}{n\omega/c} = \frac{c}{n} = \frac{c}{\sqrt{\varepsilon(\omega)}} \Rightarrow$$

$$\text{if } v > v_{ph} = \frac{c}{\sqrt{\varepsilon(\omega)}} \Rightarrow \beta > \frac{1}{\sqrt{\varepsilon}} \Rightarrow \varepsilon\beta^2 > 1$$

$$\Rightarrow \lambda^2 = \frac{\omega^2}{v^2}\left[1 - \varepsilon\beta^2\right] < 0 \Rightarrow \lambda \text{ is } \mathrm{Im}\,!$$

$$\Rightarrow \text{If } |\lambda| a \gg 1 \Rightarrow K_\nu(-i|\lambda|a) \sim e^{+i|\lambda|a}$$

$$\left(\text{at large argument } K_\nu(x) \to \sqrt{\frac{\pi}{2x}}\, e^{-x},\ x \gg 1\right)$$

$\Rightarrow$ write $\lambda = -i|\lambda|$ ("−" to have $\mathrm{Im}\,\varepsilon(\omega) > 0$, $\omega > 0$)

$$\Rightarrow K_1(\lambda^* a) K_0(\lambda a) = \frac{\pi}{2a\sqrt{\lambda\lambda^*}}\, \underbrace{e^{+i\lambda a - i\lambda a}}_{1}$$

$$\Rightarrow \lambda^* \cancel{a} \cdot \frac{\pi}{2\cancel{a}\sqrt{\lambda\lambda^*}} = \frac{\pi}{2}\sqrt{\frac{\lambda^*}{\lambda}} = i\,\frac{\pi}{2}$$

$$\Rightarrow \boxed{\left(\frac{d\mathcal{E}}{dx}\right)_{b>a} = \frac{q^2}{c^2} \int\limits_{\varepsilon\beta^2>1} d\omega \cdot \omega \left[1 - \frac{1}{\varepsilon(\omega)\beta^2}\right]}$$

Frank-
-Tamm
1937

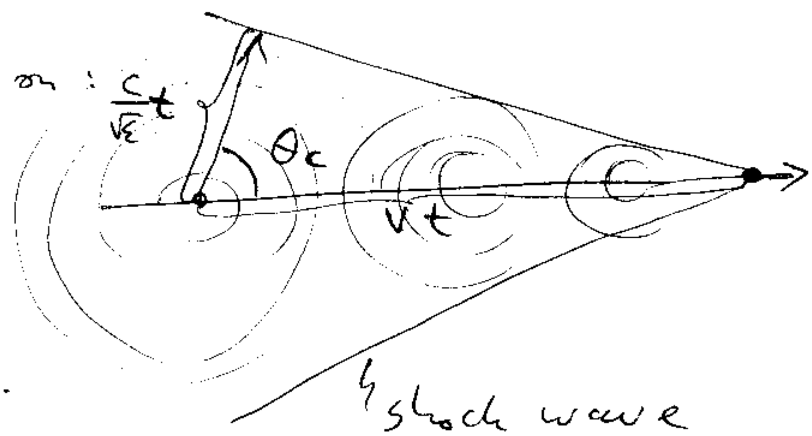
as $K_\nu(\mp i|\lambda|a) \propto e^{\pm i|\lambda|a}$ ~ no more damping

(if λ was real $\Rightarrow K_\nu(\lambda a) \propto e^{-\lambda a}$ ~ damping)

$\Rightarrow$ the radiation escapes $\Rightarrow$ Cherenkov radiation (1934)
experimentally observed first.

$\Rightarrow \left(\frac{d\mathcal{E}}{dx}\right)_{b>a}$ has the meaning of total energy emitted by Cherenkov radiation.

$\Rightarrow$ Angle of emission:



Radiation travels with velocity $\frac{c}{n} = \frac{c}{\sqrt{\varepsilon}}$.

Charge travels with velocity v

$$\Rightarrow \cos\theta_c = \frac{c/\sqrt{\varepsilon}}{v} = \frac{1}{\beta\sqrt{\varepsilon}}.$$

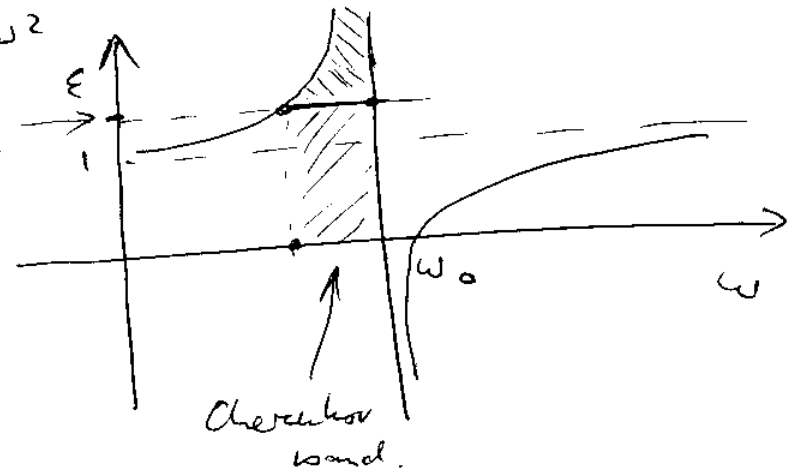
$\Rightarrow$ the faster the particle moves the large θ_c is.

$\Rightarrow$ more ~~precisely~~ rigorously one can find θ_c from calculating the direction of $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$.

<u>Cherenkov band</u> : need $\varepsilon\beta^2 > 1 \Rightarrow \varepsilon > \frac{1}{\beta^2}$.

If $\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}$ then

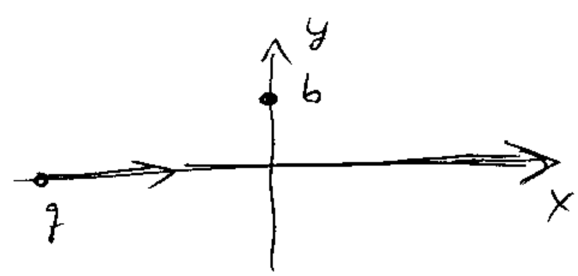
the allowed frequency region, known as Cherenkov band, is shown below. →



⇒ Cherenkov radiation is used in particle detectors to detect various particle species and/or their velocities & in cosmic rays too.

Weizsäcker - Williams Method of Virtual Quanta.

Imagine a point charge q moving with constant velocity $\vec{v}$ along the x-axis. As before we measure $\vec{E}, \vec{B}$ at point b along the y-axis:



The fields are

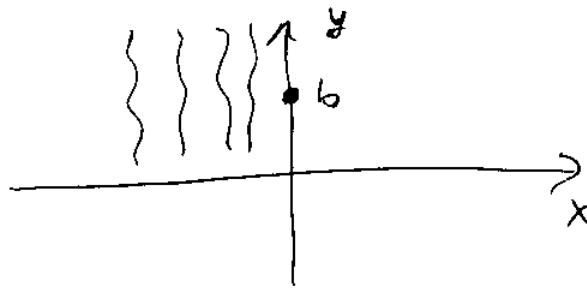
$$E_y = \frac{q\gamma b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$$B_z = \beta E_y$$

$$E_x = -\frac{q\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

(just boost from the rest frame).

$\Rightarrow$ E_y & B_z look like fields of a plane wave along the x-axis:



$\Rightarrow$ E_x field has no B-field with it $\Rightarrow$ not a plane wave

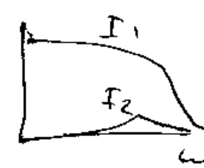
$\Rightarrow$ however its contribution is suppressed in the UR limit $\beta \to 1, \gamma \to \infty$.

$\Rightarrow$ Frequency spectrum is $\frac{dI}{d\omega} = 2\left|\sqrt{\frac{c}{4\pi}}\,\vec{E}(\omega)\right|^2 =$

$= \frac{c}{2\pi}\left[|E_y|^2 + |E_x|^2\right](\omega) \Rightarrow$ performing Fourier transforms on $\vec{E}$ & plugging the results in yields:

$$\left.\begin{matrix} \frac{dI_y}{d\omega} \\ \\ \frac{dI_x}{d\omega} \end{matrix}\right\} = \frac{1}{\pi^2}\frac{q^2}{c}\left(\frac{c}{v}\right)^2\frac{1}{b^2}\left(\frac{\omega b}{\gamma v}\right)^2 \left\{\begin{matrix} K_1^2\left(\frac{\omega b}{\gamma v}\right) \\ \\ \frac{1}{\gamma^2}K_0^2\left(\frac{\omega b}{\gamma v}\right) \end{matrix}\right.$$

$\Rightarrow \frac{dI_x}{d\omega} \ll \frac{dI_y}{d\omega}$ as $\gamma \to \infty$ $\Rightarrow$ negligible.



To calculate the spectrum integrate over b:

$$\frac{dI}{d\omega} = \int_{b_{min}}^{\infty} db \cdot 2\pi b \cdot \left[\frac{dI_y}{d\omega} + \frac{dI_x}{d\omega}\right] \Rightarrow$$

$$\frac{dI}{d\omega} = \frac{2}{\pi}\frac{q^2}{c}\left(\frac{c}{v}\right)^2\left\{x\, K_0(x)K_1(x) - \frac{v^2}{2c^2}x^2\left[K_1(x)^2 - K_0(x)^2\right]\right\}$$

where $x = \dfrac{\omega b_{min}}{\gamma v}$.

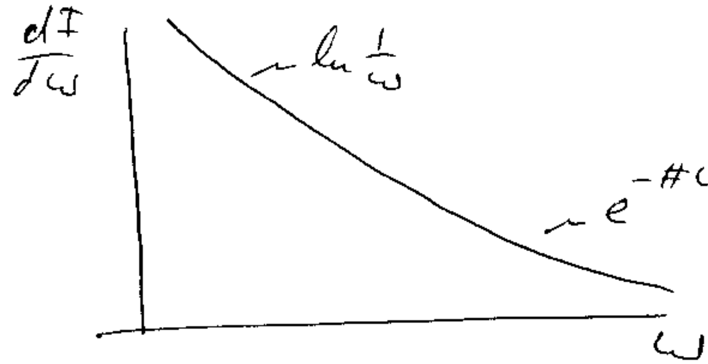
at low frequencies: $\omega \ll \dfrac{\gamma v}{b_{min}}$ $(x \ll 1) \Rightarrow$

$$\frac{dI}{d\omega} \simeq \frac{2}{\pi}\frac{q^2}{c}\left(\frac{c}{v}\right)^2\left[\ln\left(\frac{\gamma v}{\omega b_{min}}\right) - \frac{v^2}{2c^2}\right]$$

at high frequencies: $\omega \gg \dfrac{\gamma v}{b_{min}}$

$$\frac{dI}{d\omega} \propto e^{-\frac{2\omega b_{min}}{\gamma v}} \sim \text{small}$$

$\Rightarrow$ dominated by low-ω frequencies



$\Rightarrow$ One can think of incoming plane wave as of a collection of virtual quanta of light – photons. Their number spectrum is

$$\frac{dI}{d\omega}d\omega = \underbrace{\hbar\omega}\, N(\omega)\, d(\hbar\omega)$$

each of them carries energy $\hbar\omega$.

$\Rightarrow$ at $\omega \ll \dfrac{\gamma c}{b_{min}} \Rightarrow$

$$N(\omega) \simeq \frac{2}{\pi}\left(\frac{q^2}{\hbar c}\right)\left(\frac{c}{v}\right)^2 \frac{1}{\hbar\omega}\left[\ln\left(\frac{\gamma v}{\omega b_{min}}\right) - \frac{v^2}{2c^2}\right]$$

$\Rightarrow$ One can use this picture in high-energy scattering: a charged particle has a lot of virtual photons in its wave-function, which independently scatter on a target.

$\Rightarrow$ Imagine scattering of electron on a nucleus: as $N(\omega) \sim \frac{1}{\omega} \Rightarrow \frac{d\sigma}{d\omega} \propto \frac{1}{\omega}$

($\frac{d\sigma}{d\omega}$ is the scattering x-section,

$$\frac{d\sigma}{d\omega} \propto N(\omega)\cdot\hat{\sigma} \propto \frac{1}{\omega} \Bigg).$$

$\boxed{\frac{d\sigma}{d\omega} \propto \frac{1}{\omega}}$ is a typical bremsstrahlung spectrum

$\Rightarrow$ The same picture applies to <u>strong interactions</u>: a proton consists of quarks and gluons, which, at high energy, can be viewed as WW quanta in its wave function!