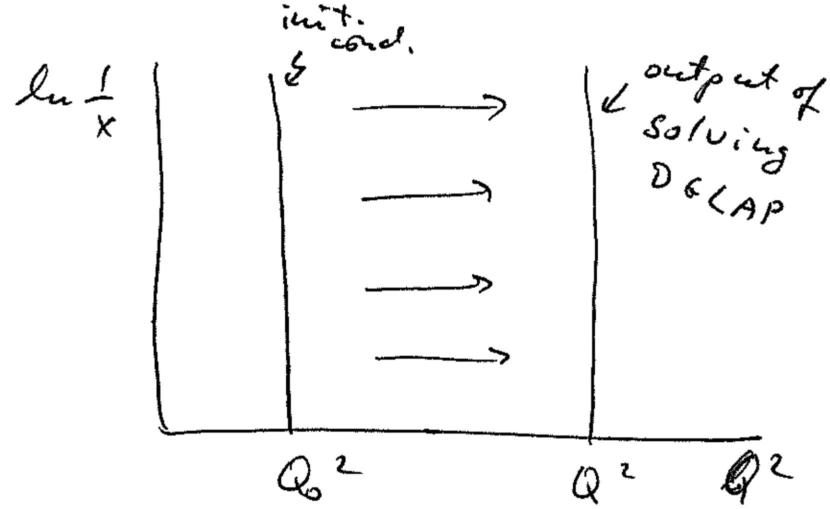
Last time talked about DGLAP equation

~ usually one sets initial conditions at some $Q^2 = Q_0^2$ & "evolves" the PDFs with DGLAP to $Q^2 > Q_0^2$.



DGLAP at small x

Gluons dominate at small x ⟹ drop quarks

$$\frac{\partial}{\partial \ln Q^2} G(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dx_1}{x_1} P_{GG}\left(\frac{x}{x_1}\right) G(x_1, Q^2)$$

Def. moments: $$G_n(Q^2) = \int_0^1 dx \cdot x^{n-1} \cdot G(x, Q^2)$$

anomalous dimension: $$\gamma_{GG}^{(n)} = \int_0^1 dz \cdot z^{n-1} P_{GG}(z)$$

$$P_{GG}(z) \simeq \frac{2N_c}{z} \quad \text{at small-}x$$

In moments space, the solution is easy:

$$G_n(Q^2) = e^{\int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \frac{\alpha_s(Q'^2)}{2\pi} \gamma_{GG}^{(n)}} G_n(Q_0^2)$$

such that

$$\boxed{G(x, Q^2) = \int \frac{dn}{2\pi i} \, x^{-n} \, e^{\frac{\gamma_{GG}^{(n)}}{2\pi\beta_2} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)} G_n(Q_0^2)}$$

Oscillations are not there only at the saddle point:

$$\frac{d}{dn}\left[n \ln\frac{1}{x} + \frac{N_c}{n-1}\,\frac{1}{\pi\beta_2}\,\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]\Bigg|_{n=n_0} = 0$$

$$\ln\frac{1}{x} - \frac{N_c}{(n_0-1)^2}\,\frac{1}{\pi\beta_2}\,\ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right) = 0$$

$$n_0 - 1 = \pm\sqrt{\frac{N_c}{\pi\beta_2}\ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right)\frac{1}{\ln\frac{1}{x}}}$$

"+" dominates (gives larger contribution).
to $(n_0-1)\ln\frac{1}{x}$

To estimate the integral we define the power of the exponent

$$P(n) = n\ln\frac{1}{x} + \frac{N_c}{n-1}\,\frac{1}{\pi\beta_2}\,\ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right)$$

and expand

$$P(n) \approx P(n_0) + \frac{1}{2}(n-n_0)^2\,P''(n_0)$$

where
$$P''(n_0) = +\frac{2N_c}{(n_0-1)^3}\,\frac{1}{\pi\beta_2}\,\ln\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2} = \frac{2N_c}{\pi\beta_2}\ln\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\,.$$

$$\left(\frac{\pi b}{N_c}\right)^{3/2}\left[\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]^{-3/2}\ln^{3/2}\frac{1}{x} = 2\left(\frac{\pi\beta_2}{N_c}\right)^{1/2}\ln^{3/2}\frac{1}{x}\cdot\left[\ln\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right]^{-1/2}.$$

$$P(n_0) = \ln\frac{1}{x} + 2\sqrt{\frac{N_c}{\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\ln\frac{1}{x}}\,.$$

$$\int \frac{dn}{2\pi i}\, e^{P(n_0) + \frac{1}{2}(n-n_0)^2 P''(n_0)} = \left| n - n_0 = i\xi \right. = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\, e^{P(n_0) - \frac{1}{2}\xi^2 P''(n_0)} =$$

$$= \frac{1}{2\pi}\, e^{P(n_0)} \sqrt{\frac{2\pi}{P''(n_0)}} = \frac{e^{P(n_0)}}{\sqrt{2\pi P''(n_0)}}$$

we obtain

$$x G(x, Q^2) = G_{n_0}(Q_0^2) \cdot e^{2\sqrt{\frac{N_c}{\pi \beta_2} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right) \ln \frac{1}{x}}} \cdot \frac{1}{\sqrt{4\pi}}$$

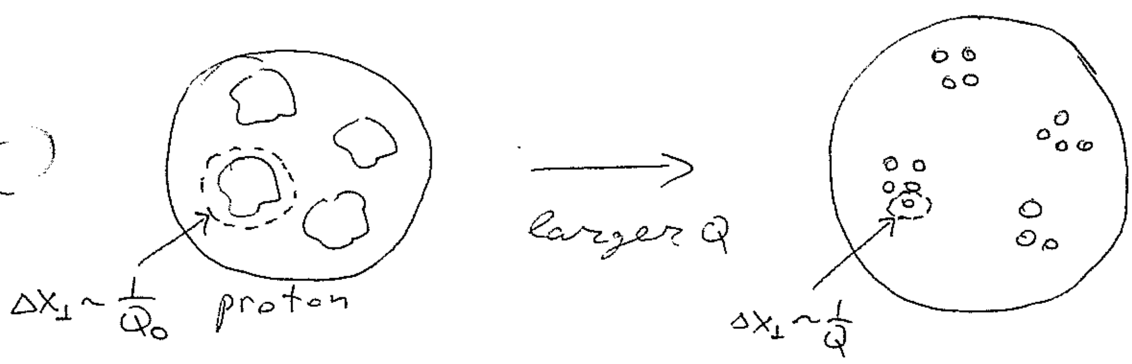
$$\cdot \left(\frac{N_c}{\pi \beta_2}\right)^{1/4} \ln^{-3/4} \frac{1}{x} \left[\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]^{1/4}$$

also note that xG grows with Q^2

Therefore, $xG \sim e^{2\sqrt{\frac{N_c}{\pi \beta_2} \ln \frac{1}{x} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)}}$

xG grows at small-x, slower than a power of x but faster than any power of $\ln \frac{1}{x}$. $\Rightarrow$ may explain rise of xG at small-x...

How DGLAP works: we increase Q/resolution, see more partons



Renormalization Groups.

A Note on the Saddle Point Method

(a.k.a. the Method of Steepest Descent)

$$I(\lambda) = \int_C dz\, g(z)\, e^{\lambda f(z)}$$

$f(z)$, $g(z)$ analytic functions

$\lambda \gg 1$ ~ large parameter

(i) Find a point z_0 such that $\boxed{f'(z=z_0)=0}$.

(ii) Deform the contour C to go through z_0 along the $\operatorname{Im} f(z) = \operatorname{Im} f(z_0)$ line.
(Line of steepest descent.)

(iii) Evaluate the resulting integral. In most practical applications one can approximate $f(z) \simeq f(z_0) + \frac{1}{2} f''(z_0)(z-z_0)^2$ such that

$$I \simeq g(z_0)\, e^{\lambda f(z_0)} \int dz\, e^{\frac{\lambda}{2} f''(z_0)(z-z_0)^2}$$

for $\lambda \gg 1$.