Last time | **Functional Integral Quantization** (cont'd)

Path Integral Quantum Mechanics (cont'd)

Non-relativistic 1-particle QM: $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$

$[\hat{q}, \hat{p}] = i\hbar$, $\psi(q,t) = {}_S\langle q(t)|\psi(t)\rangle_S$ (wave function)

$$\to \psi(q,t) = \int_{-\infty}^{\infty} dq' \langle q(t)| e^{-\frac{i}{\hbar}\hat{H}(t-t')} |q'(t')\rangle \psi(q',t')$$

Def. Time-evolution (Feynman) kernel:

$$U(q,t;q',t') \equiv {}_S\langle q(t)| e^{-\frac{i}{\hbar}\hat{H}(t-t')} |q'(t')\rangle_S .$$

We showed that

$$U(q,t;q',t') = \lim_{N\to\infty} \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N-1} \frac{dq_i\, dp_i}{2\pi\hbar}\right] \frac{dp_N}{2\pi\hbar}\, e^{\frac{i}{\hbar}\delta t \sum_{j=1}^{N}\left[p_j \frac{q_j - q_{j-1}}{\delta t} - H\right]}$$

& denoted this object by

$$U(q,t;q',t') = \int [\mathcal{D}q\, \mathcal{D}p]\, e^{\frac{i}{\hbar}\int_{t'}^{t} dt''\left[p(t'')\dot{q}(t'') - H(p(t''), q(t''))\right]}$$

path integral.

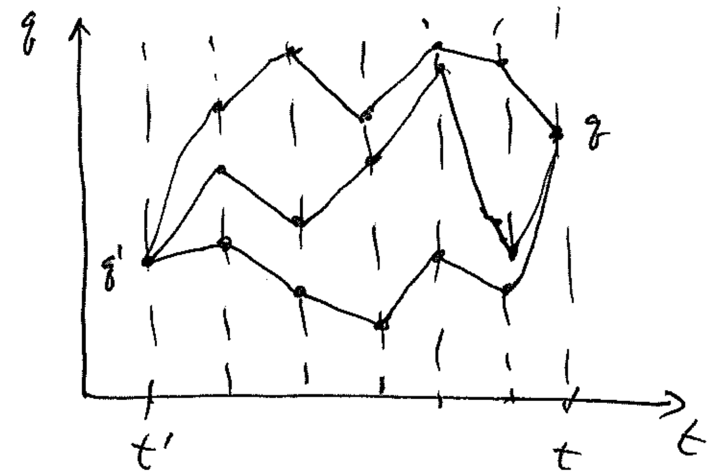
Integrating out $\mathcal{D}p$ we got

$$U(q,t;q',t') = \lim_{N\to\infty} \int_{-\infty}^{\infty}\left[\prod_{i=1}^{N-1} dq_i\right]\left[\frac{m}{2\pi\hbar i\delta t}\right]^{N/2} e^{\frac{i}{\hbar}\delta t \sum_{j=1}^{N} L(q_j, \dot{q}_j)}$$

which we denoted by

$$\boxed{\begin{aligned} U(q,t;q',t') &= \mathcal{N}\int [\mathcal{D}q]\, e^{\frac{i}{\hbar}\int_{t'}^{t} dt''\, L(q(t''),\,\dot{q}(t''))} \\ &= \mathcal{N}\int [\mathcal{D}q]\, e^{\frac{i}{\hbar} S(q,t;q',t')} \end{aligned}}$$

integral over all paths:



<u>Example</u> Free particle, $V(q) = 0$

$$U(q,t;q',t') = \lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar\,\delta t}\right]^{N/2}\int_{-\infty}^{\infty}\left[\prod_{i=1}^{N-1} dq_i\right] e^{\frac{i}{\hbar}\delta t\,\frac{m}{2}\sum_{j=1}^{N}\frac{(q_j-q_{j-1})^2}{\delta t^2}}$$

$\Rightarrow$ integrated out q_1

Last time | Harmonic oscillator:

$$\boxed{\mathcal{U}_{ho}(q_f, t_f; q_i, t_i) = \sqrt{\frac{m}{2\pi i \hbar T}} \sqrt{\frac{\omega T}{\sin \omega T}} \, e^{\frac{i}{\hbar} S_{cl}}}$$

Time-ordered product: $\hat{q}_H(t) = e^{\frac{i}{\hbar}\hat{H}t} \, \hat{q}_S \, e^{-\frac{i}{\hbar}\hat{H}t}$

$$|q, t\rangle_H = e^{\frac{i}{\hbar}\hat{H}t} |q(t)\rangle_S$$

$$\boxed{\mathcal{U}(q_f, t_f; q_i, t_i) = {}_H\langle q_f, t_f | q_i, t_i\rangle_H} = N \int [\mathcal{D}q] \, e^{\frac{i}{\hbar} S}.$$

$$\boxed{{}_H\langle q_f, t_f | T \hat{q}_H(t_2) \hat{q}_H(t_1) | q_i, t_i\rangle_H = N \int [\mathcal{D}q] \, q(t_2) q(t_1) e^{\frac{i}{\hbar} S}}$$

$$\boxed{{}_H\langle q_f, t_f | T \hat{q}_H(t_1) \ldots \hat{q}(t_N) | q_i, t_i\rangle_H = N \int [\mathcal{D}q] \, q(t_1) \ldots q(t_N) \cdot e^{\frac{i}{\hbar} S}}$$

Vacuum-to-vacuum tr. amplitude:

$$Z[j] \propto {}^{j}\langle 0, +\infty | 0, -\infty\rangle^{j}$$

$$L \to L + \hbar j q$$

$$\boxed{Z[j] = \int [\mathcal{D}q] \, e^{\frac{i}{\hbar} \int_{-\infty(1-i\delta)}^{\infty(1+i\delta)} dt \, [L + \hbar j(t) q(t)]}}$$

Last time | Functional Quantization of the Scalar Field Theory (cont'd)

By analogy with QM, we introduced generating functional for Green functions:

$$\boxed{Z[j] = \int \mathcal{D}\varphi \, e^{i\int d^4x\,[\mathcal{L} + j(x)\varphi(x)]}}$$

where $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + \mathcal{L}_{int}$.

Any n-point Green function is then

$$\boxed{\langle\Psi_0|\,T\{\varphi_H(x_1)\ldots\varphi_H(x_n)\}\,|\Psi_0\rangle = \frac{\int\mathcal{D}\varphi\;\varphi(x_1)\ldots\varphi(x_n)\,e^{i\int d^4x\,\mathcal{L}}}{\int\mathcal{D}\varphi\;e^{i\int d^4x\,\mathcal{L}}}}$$

or, equivalently,

$$\boxed{\langle\Psi_0|\,T\{\varphi_H(x_1)\ldots\varphi_H(x_n)\}\,|\Psi_0\rangle = (-i)^n\frac{1}{Z[j=0]}\,\frac{\delta^n Z[j]}{\delta j(x_1)\ldots\delta j(x_n)}\bigg|_{j=0}}$$

Free Scalar Theory

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \Rightarrow \boxed{Z_0[j] = \int\mathcal{D}\varphi\cdot e^{i\int d^4x\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2-i\varepsilon}{2}\varphi^2 + j\varphi\right]}}.$$

We showed that

$$\boxed{\int (dx)\, e^{-\frac{1}{2}x^T A x} = \frac{1}{\sqrt{\det A}}}, \quad (dx) = \frac{d^n x}{(2\pi)^{n/2}}, \quad \begin{pmatrix}A \sim \text{symmetric real}\\ \Rightarrow \text{invertible}\end{pmatrix}$$

$\Rightarrow$ by defining φ_0 using $\left(\Box + m^2 - i\varepsilon\right)\varphi_0 = j$ & the above formula we proved that

$$Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}} \, e^{\frac{i}{2}\int d^4x \, j \cdot \varphi_0}$$

where $\hat{D} = \Box + m^2 - i\varepsilon$

$$\Rightarrow \boxed{Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}} \, e^{-\frac{1}{2}\int d^4x \, d^4y \, j(x) \, D_F(x-y) \, j(y)}}$$

where $\boxed{D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon}}$.

(Feynman propagator)