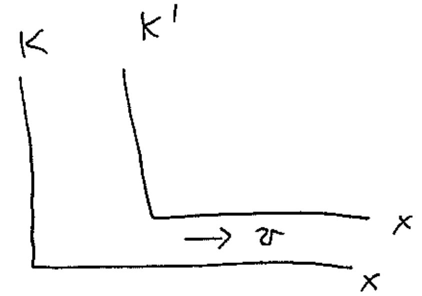
Classical Field Theory

4-vectors, notations



defining $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

write Lorentz transformation

as
$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

$\beta = v/c$

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$

Definition: a 4-vector $A^\mu = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$ is an object

which under Lorentz transformation transforms

as
$$\begin{pmatrix} A'^0 \\ A'^1 \\ A'^2 \\ A'^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda^\mu{}_\nu} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}.$$

(example: x^μ is a contravariant 4-vector)

$A'^\mu = \Lambda^\mu{}_\nu A^\nu$

$\Rightarrow$ A^μ is a contravariant vector: $A'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} A^\nu$

B_μ is a covariant vector: $B'_\mu = \dfrac{\partial x^\nu}{\partial x'^\mu} B_\nu$

$\Rightarrow$ $\dfrac{\partial \varphi}{\partial x^\mu} \equiv \partial_\mu \varphi$ with φ scalar field is a covariant vector as $\frac{\partial \varphi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \varphi}{\partial x^\nu}$.

Tensors: $A^\mu B^\nu$ ~ contravariant, $A_\mu B_\nu$ ~ covariant (rank 2), can have higher ranks.

Def. Scalar (inner) product of 2 vectors is $A_\mu B^\mu$. (assume summation).

It is Lorentz-invariant: $A'_\mu B'^\mu = \frac{\partial x^\alpha}{\partial x'^\mu} A_\alpha \frac{\partial x'^\mu}{\partial x^\beta} B^\beta$

$$= \frac{\partial x^\alpha}{\partial x^\beta} A_\alpha B^\beta = \delta^\alpha_\beta A_\alpha B^\beta = A_\alpha B^\alpha.$$

Def. The interval $ds^2 \left(= dx_\mu dx^\mu\right) = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$

It's a Lorentz-invariant too.

Def. The metric tensor $g_{\mu\nu}$ is defined by

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}$$

In our Minkowski space (throughout the course we'll use this notation) $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta_{\mu\nu}$

$dx_\mu dx^\mu$ ~ also a Lorentz-scalar $\Rightarrow dx_\mu = g_{\mu\nu} dx^\nu$

$\Rightarrow g_{\mu\nu}$ lowers & raises indeces!

Example: $x^\mu = (ct, \vec{x}) \Rightarrow x_\mu = g_{\mu\nu} x^\nu = (ct, -\vec{x})$

contravariant covariant

In general $A_\mu = g_{\mu\nu} A^\nu$, $A^\mu = g^{\mu\nu} A_\nu$

where $g^{\mu\nu}$ is defined by requiring that

$\boxed{g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu}$: if that is true $\Rightarrow$ start

with $A_\mu = g_{\mu\nu} A^\nu \Rightarrow g^{\alpha\mu} A_\mu = g^{\alpha\mu} g_{\mu\nu} A^\nu =$

$= \delta^\alpha{}_\nu A^\nu = A^\alpha \Rightarrow A^\alpha = g^{\alpha\mu} A_\mu$.

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta^{\mu\nu} \text{ too.} \quad (= g_{\mu\nu})$$

Def. $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$, $\partial^\mu \equiv \dfrac{\partial}{\partial x_\mu} \Rightarrow \partial_\mu \varphi$ is a covariant vector,

$\partial^\mu \varphi$ is a contravariant vector. (check!)

$\partial_\mu A^\mu$ is a Lorentz-invariant.

$\partial_\mu \partial^\mu = \dfrac{1}{c^2} \dfrac{\partial^2}{\partial t^2} - \vec{\nabla}^2$ is also Lorentz-invariant.

Examples: other important 4-vectors are

$$p^\mu = \begin{pmatrix} \mathcal{E}/c \\ \vec{p} \end{pmatrix}, \quad p_\mu = \begin{pmatrix} \mathcal{E}/c \\ -\vec{p} \end{pmatrix} \Rightarrow p_\mu p^\mu = \left(\frac{\mathcal{E}}{c}\right)^2 - \vec{p}^{\,2} = m^2 c^2.$$

$A^\mu = (\Phi, \vec{A})$ in E&M, Φ ~ electric potential,

$\vec{A}$ - vector potential.

$J^\mu = (c\rho, \vec{J})$ with ρ the charge density, $\vec{J}$ the current density.

Notations from now on $\boxed{c = 1}$ and $\boxed{\hbar = 1}$

"natural units".

$\Rightarrow$ mass, momentum, energy are measured in

the same units (eV, keV, MeV, GeV, ...)

$$1\,\text{eV} = 1.6 \times 10^{-19}\,\text{J}.$$

distances, time are measured in femto-

-meters aka fermis (fm):

$$1\,\text{fm} \simeq 5\,\text{GeV}^{-1}$$

$1\,\text{GeV} = 10^9\,\text{eV}$, 1 femto-meter $:= 10^{-15}\,\text{m}. = 1\,\text{fm}.$

proton's mass $m_p = 0.938\,\text{GeV} \simeq 1\,\text{GeV}$.

electron's mass $m_e = 0.511\,\text{MeV} = 0.5 \times 10^{-3}\,\text{GeV}$

Classical Scalar Field Theory (real field)

$\varphi(x^\mu) = \varphi(x^0, \vec{x})$ ~ a function of space-time points x^μ

(example ~ temperature field $T(t, \vec{x})$)

in Classical Mechanics one has point particles $i = 1, \ldots, N$ with the Lagrangian $L(q_i, \dot{q}_i, t)$

and the action $S = \int dt \, L(q_i, \dot{q}_i, t)$

q_i ~ degrees of freedom (e.g. particle coordinates)

$\dot{q}_i = \frac{dq_i}{dt}$ ~ generalized velocities

Now, instead of discrete point particles we have a field $\varphi(\vec{x}, t)$ =>

Classical Mechanics		Classical Field Theory
q_i	$\longrightarrow$	$\varphi(x^0, \vec{x})$
i	$\longrightarrow$	$\vec{x}, t$
$\dot{q}_i$	$\longrightarrow$	$\partial_\mu \varphi$, $\mu = 0, 1, 2, 3$
$L(q_i, \dot{q}_i, t)$	$\longrightarrow$	$\int d^3x \, \mathcal{L}(\varphi, \partial_\mu \varphi)$

$\mathcal{L}$ is <u>Lagrangian</u> <u>density</u>. (usually called the Lagrangian)

The action is $S = \int dt\, L = \underbrace{\int dt\, d^3x}_{d^4x} \mathcal{L}(\varphi, \partial_\mu \varphi)$ (remember $c=1$)

S is a Lorentz-scalar (better be, physics is Lorentz-invariant)

What about $d^4x = dx^0 dx^1 dx^2 dx^3$? Remember that

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$ with $\Lambda^\mu{}_\nu = \dfrac{\partial x'^\mu}{\partial x^\nu}$ a matrix of L. tr.

$\Rightarrow d^4x' = |\underbrace{\det \Lambda}_{\text{Jacobian}}|\, d^4x$.

Now, $\det \Lambda = \det \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \gamma^2(1-\beta^2) = 1$ (true in general)

$\Rightarrow d^4x' = d^4x \Rightarrow d^4x$ is a Lorentz-scalar

$\Rightarrow$ <u>$\mathcal{L}$ is a Lorentz-scalar!</u>

Just like in classical mechanics, in classical field theory dynamics is given by the <u>least action principle</u>; field φ is determined by requiring that S is stationary with respect to small perturbation around φ: $S[\varphi + \delta\varphi] = S[\varphi] + o(\delta\varphi^2)$.