Last time | **Classical Fied Theory (cont'd)**

Classical Scalar Field Theory (cont'd)

Classical Mechanics	Classical Field Theory
q_i	$\varphi(x)$
i	x^μ
$\dot{q}_i$	$\partial_\mu \varphi$
$L(q_i, \dot{q}_i)$	$\int d^3x \, \mathcal{L}(\varphi, \partial_\mu \varphi)$
$S = \int dt \, L(q_i, \dot{q}_i)$	$S = \int d^4x \, \mathcal{L}(\varphi, \partial_\mu \varphi)$

S = action , $\mathcal{L}$ = Lagrangian density

S = boost invariant , d^4x = boost invariant

$\Rightarrow \mathcal{L}$ is boost invariant.

Dynamics in classical physics is determined by the least action principle: $\boxed{\delta S = 0}$.

$$0 = \delta S = \int d^4x \left[\frac{\delta \mathcal{L}}{\delta \varphi} \delta\varphi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right] =$$

$$= (\text{as } \delta \partial_\mu \varphi = \partial_\mu \delta\varphi \Rightarrow \text{parts}) = \int d^4x \left[\frac{\delta \mathcal{L}}{\delta\varphi} \delta\varphi - \right.$$

$$\left. - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \right) \delta\varphi \right] + \underbrace{\text{surface term}}_{=0}$$

$$\Rightarrow 0 = \int d^4x \, \delta\varphi \left[\frac{\delta \mathcal{L}}{\delta\varphi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \right) \right] \quad \text{for any } \delta\varphi$$

$$\Rightarrow \boxed{\frac{\delta \mathcal{L}}{\delta\varphi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \right) = 0}$$

Euler - Lagrange equations (aka equations of motion) for field φ.
(EOM)

Now, $\varphi(x)$ is a scalar field $\Rightarrow$ it is Lorentz-inv.,

which means that : $\varphi(x) \to \varphi'(x') = \varphi(x)$

$\Rightarrow$ as $x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow x' = \Lambda \cdot x \Rightarrow \varphi'(x) = \varphi(\Lambda^{-1} x)$.

Lagrangian density for massive free scalar field:

$$\boxed{\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{m^2}{2} \varphi^2}$$

(i) $\mathcal{L}$ is Lorentz-inv.

(ii) $\mathcal{L}$ is $\mathcal{O}(\varphi^2)$

(iii) No ∂^3 or higher (causality)

EOM: $\dfrac{\delta \mathcal{L}}{\delta\varphi} = -m^2 \varphi$; $\dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} = \partial^\mu \varphi \Rightarrow$

$$\Rightarrow \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \right) = \partial_\mu \partial^\mu \varphi \Rightarrow -m^2\varphi - \partial_\mu \partial^\mu \varphi = 0$$

$$\Rightarrow \boxed{\left[\partial_\mu \partial^\mu + m^2\right]\varphi = 0}$$ Klein-Gordon equation

or $\left[\Box + m^2\right]\varphi = 0$

To solve K-G equation write $\varphi(x) = \int d^4k\, e^{-ik\cdot x}\,\tilde{\varphi}(k)$

with $k\cdot x = k_\mu x^\mu = k^0 x^0 - \vec{k}\cdot\vec{x}$.

$$\left[\Box + m^2\right]\varphi = \int d^4k\, \tilde{\varphi}(k)\left(\Box + m^2\right)e^{-ik\cdot x} = \int d^4k\, \tilde{\varphi}(k)\cdot e^{-ik\cdot x}$$

$\cdot\left[-k^2 + m^2\right] = 0$ with $k^2 = k_\mu k^\mu = (k^0)^2 - (\vec{k})^2$.

$\Rightarrow \left[k^2 - m^2\right]\tilde{\varphi} = 0 \Rightarrow$ as $\tilde{\varphi} \neq 0 \Rightarrow k^2 = m^2$ or

$\mathcal{E}_k^2 - \vec{k}^2 = m^2 \Rightarrow \mathcal{E}_k = \pm\sqrt{\vec{k}^2 + m^2} \Rightarrow$ define $\mathcal{E}_k = \sqrt{\vec{k}^2 + m^2}$

$$\Rightarrow \varphi(x) = \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k}\left[a_{\vec{k}}\, e^{-i\mathcal{E}_k t + i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^{*}\, e^{i\mathcal{E}_k t - i\vec{k}\cdot\vec{x}}\right]$$

most general solution.

~~Canonical Quantization~~

~~In your QM class you must have seen that if we treat K-G equation as the equation for a single-particle wave function $\varphi(x)$ (just like~~

One in general would write

$$\tilde{\varphi}(k) \propto (2\pi)\, \delta(k^2 - m^2)\, \theta(k^0)$$

pick "physical" (positive energy) solution

$$\Rightarrow \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{\varphi}(k) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} (2\pi)\, \delta(k^2-m^2)\, \theta(k^0).$$

$$\cdot a_{\vec{k}} = \int \frac{d^3k}{(2\pi)^3} dk^0\, e^{-ik^0 t + i\vec{k}\cdot\vec{x}}\, \delta((k^0)^2 - \vec{k}^2 - m^2).$$

coefficient

$$\cdot \theta(k^0)\, a_{\vec{k}} = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathcal{E}_k t + i\vec{k}\cdot\vec{x}} \frac{1}{2\mathcal{E}_k} a_{\vec{k}} =$$

$$= \int \underbrace{\frac{d^3k}{2\mathcal{E}_k (2\pi)^3}}_{\substack{\text{boost invariant} \\ \text{phase space measure}}} e^{-i\mathcal{E}_k t + i\vec{k}\cdot\vec{x}}\, a_{\vec{k}}$$

$\Rightarrow$ to get full real φ add c.c. $\Rightarrow$

$$\Rightarrow \varphi = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \left[a_{\vec{k}}\, e^{-i\mathcal{E}_k t + i\vec{k}\cdot\vec{x}} + a^*_{\vec{k}}\, e^{i\mathcal{E}_k t - i\vec{k}\cdot\vec{x}} \right].$$

<u>problem:</u> have negative energy modes, having $-\mathcal{E}_k$.
If φ is a wave function ~ have negative energy states ~ bad.

Def. Canonical momentum field

$$\boxed{\pi(x) \equiv \frac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)}}$$

$$\Rightarrow \quad \boxed{\mathcal{H} = \pi(x)\,\dot{\varphi}(x) - \mathcal{L}}$$ Hamiltonian density

Real scalar field: $\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2$

$$\pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}} = \dot{\varphi} \quad \Rightarrow \quad \mathcal{H} = \pi \underset{\pi}{\underset{\shortparallel}{\dot{\varphi}}} - \mathcal{L} = \pi^2 - \frac{1}{2}\pi^2 +$$

$$+ \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 \quad \Rightarrow$$

$$\boxed{\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2}$$

$\mathcal{H} \geq 0 \Rightarrow$ energy of the <u>field</u> is non-negative.

Conservation Laws & Noether's Theorem.

- **Noether's theorem** | Every continuous symmetry (of S) gives a conservation law.

If $S \to S' = S$ when $\phi \to \phi'$, $x^\mu \to x'^\mu$ $\Rightarrow$ there exists one or more conserved quantities.

Example 1 | Consider complex (!) scalar field $\phi(x)$ with $\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 |\phi|^2$, $\left(\phi = \frac{\overbrace{\phi_1 + i\phi_2}^{\text{real}}}{\sqrt{2}}\right)$

It is invariant under $\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$ with α a real constant. ($U(1)$ symmetry group)

$$0 \overset{\mathcal{L}\text{ is inv.}}{=} \delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \phi}\delta\phi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \frac{\delta \mathcal{L}}{\delta \phi^*}\delta\phi^*$$

$$+ \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)}\delta(\partial_\mu \phi^*) = \underbrace{\left[\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right)\right]}_{=0}\delta\phi +$$

$$+ \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta\phi\right) + \underbrace{\left[\frac{\delta \mathcal{L}}{\delta \phi^*} - \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)}\right)\right]}_{=0}\delta\phi^* +$$

$$+ \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)}\delta\phi^*\right)$$

$[\dots] = 0$ by Euler-Lagrange equations

$$\Rightarrow \quad 0 = \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta\phi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)} \delta\phi^* \right]$$

Infinitesimal transform: $\phi \to e^{i\delta\alpha}\phi \simeq (1+i\delta\alpha)\phi$

$$\Rightarrow \quad \delta\phi = i\,\delta\alpha\,\phi \quad ; \quad \delta\phi^* = -i\,\delta\alpha\,\phi^*$$

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = \partial^\mu \phi^* \quad ; \quad \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)} = \partial^\mu \phi$$

$$\Rightarrow \quad 0 = i\,\delta\alpha\,\partial_\mu \underbrace{\left[\phi\,\partial^\mu \phi^* - \phi^*\,\partial^\mu \phi \right]}_{\text{call this } j^\mu}$$

$\Rightarrow$ $\boxed{j^\mu \overset{\nabla}{=} i\left[\phi\,\partial^\mu\phi^* - \phi^*\,\partial^\mu\phi\right]}$ is a <u>conserved current</u>

as $\partial_\mu j^\mu = 0$.

In general if $S \to S' = S$ under $\phi \to \phi'$ $\Rightarrow$ as S is inv.

$$\Rightarrow \mathcal{L} \to \mathcal{L}' = \mathcal{L} + \delta\alpha \cdot \underbrace{\partial_\mu J^\mu}$$

4- divergence ~ surface term in S.

$\Rightarrow$ straightforward to find J^μ

(if $\mathcal{L} = \mathcal{L}'$ $\Rightarrow$ J^μ is conserved)