Last time | We considered scalar field theory with the Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2$

Used equations of motion $\boxed{\frac{\delta\mathcal{L}}{\delta\varphi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right) = 0}$

to obtain Klein-Gordon equation

$$\boxed{\left[\Box + m^2\right]\varphi(x) = 0.}$$

We then solved the KG equation to obtain

$$\boxed{\varphi(x) = \int\frac{d^3k}{(2\pi)^3\,2E_k}\left[a_{\vec{k}}\,e^{-iE_kt+i\vec{k}\cdot\vec{x}} + a^*_{\vec{k}}\,e^{iE_kt-i\vec{k}\cdot\vec{x}}\right].}$$

Conservation Laws & Noether's Theorem (cont'd)

every continuous symmetry of S

$\Updownarrow$

conservation law

Example 1 | $\mathcal{L} = \partial_\mu\varphi\,\partial^\mu\varphi^* - m^2|\varphi|^2$

$\varphi = \dfrac{\varphi_1 + i\varphi_2}{\sqrt{2}}$, $\varphi_1, \varphi_2 \sim$ real scalar fields

$\mathcal{L}$ is invariant under $\varphi(x) \to e^{i\alpha}\varphi(x)$,

α = real constant $\Rightarrow$ showed that

$$0 = \delta \mathcal{L} = \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \delta\varphi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi^*)} \delta\varphi^* \right]$$

$$\varphi \to e^{i\alpha}\varphi \simeq (1+i\alpha)\varphi \Rightarrow \delta\varphi = i\alpha\varphi, \quad \delta\varphi^* = -i\alpha\varphi.$$

Conservation Laws & Noether's Theorem.

<u>Noether's theorem</u> | Every continuous symmetry (of S) gives a conservation law.

If $S \to S' = S$ when $\phi \to \phi'$, $x^\mu \to x'^\mu$ $\Rightarrow$ there exists one or more conserved quantities.

<u>Example 1</u> | Consider complex (!) scalar field $\phi(x)$ with

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 |\phi|^2, \quad \left(\phi = \frac{\overbrace{\phi_1 + i\phi_2}^{\text{real}}}{\sqrt{2}} \right)$$

It is invariant under $\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$ with α a real constant. ($U(1)$ symmetry group)

$$0 \overset{\mathcal{L}\text{ is inv.}}{=} \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\delta\mathcal{L}}{\delta\phi^*}\delta\phi^*$$

$$+ \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\delta(\partial_\mu\phi^*) = \underbrace{\left[\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\right]}_{=0}\delta\phi +$$

$$+ \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right) + \underbrace{\left[\frac{\delta\mathcal{L}}{\delta\phi^*} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\right)\right]}_{=0}\delta\phi^* +$$

$$+ \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\delta\phi^*\right)$$

$[\ldots] = 0$ by Euler-Lagrange equations

$$\Rightarrow \quad 0 = \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta\phi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)} \delta\phi^* \right]$$

Infinitesimal transform: $\phi \to e^{i\delta\alpha}\phi \simeq (1+i\delta\alpha)\phi$

$$\Rightarrow \quad \delta\phi = i\,\delta\alpha\,\phi \quad ; \quad \delta\phi^* = -i\,\delta\alpha\,\phi^*$$

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = \partial^\mu \phi^* \quad ; \quad \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)} = \partial^\mu \phi$$

$$\Rightarrow \quad 0 = i\,\delta\alpha\,\partial_\mu \underbrace{\left[\phi\,\partial^\mu \phi^* - \phi^*\,\partial^\mu \phi \right]}_{\text{call this } j^\mu}$$

$$\Rightarrow \quad \boxed{j^\mu \overset{\nabla}{=} i\left[\phi\,\partial^\mu\phi^* - \phi^*\,\partial^\mu\phi\right]}$$ is a <u>conserved current</u>

as $$\partial_\mu j^\mu = 0 .$$

In general if $S \to S' = S$ under $\phi \to \phi'$ $\Rightarrow$ as S is inv.

$$\Rightarrow \mathcal{L} \to \mathcal{L}' = \mathcal{L} + \delta\alpha \cdot \underbrace{\partial_\mu J^\mu}_{\text{4-divergence} \,\sim\, \text{surface term in } S.}$$

$\Rightarrow$ straightforward to find J^μ

(if $\mathcal{L} = \mathcal{L}'$ $\Rightarrow$ J^μ is conserved)

Example 2 | Imagine a theory with

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi) \qquad (\text{no } x^\mu\text{-dependence in } \mathcal{L})$$

Imagine an infinitesimal space-time shift:

$$x^\mu \to x^\mu - \delta a^\mu = x'^\mu \Rightarrow x^\mu = x'^\mu + \delta a^\mu$$

$$\Rightarrow \phi(x) \xrightarrow{\phi \text{ is inv.}} \underbrace{\phi(x'^\mu + \delta a^\mu)}_{=\,\phi'(x')} \simeq \phi(x'^\mu) + \delta a^\mu \partial_\mu \phi(x')$$

$$\delta\mathcal{L} = \frac{\delta \mathcal{L}}{\delta\phi}\delta\phi + \left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right) = \underbrace{\left[\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\right]}_{=\,0 \ (\text{EOM})}\delta\phi$$

$$+ \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right) = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right)$$

$$\Rightarrow \delta\mathcal{L} = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta a^\nu \partial_\nu \phi\right).$$

On the other hand $\mathcal{L}$ is also a scalar $\Rightarrow \mathcal{L} = \mathcal{L}(x) \Rightarrow$

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \delta a^\mu \partial_\mu \mathcal{L} \Rightarrow \delta\mathcal{L} = \delta a^\nu \partial_\mu(\delta^\mu_\nu \mathcal{L})$$

Equating two $\delta\mathcal{L}$'s we get

$$\delta a^\nu \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu \mathcal{L}\right] = 0$$

Def. Energy-momentum tensor

$$\boxed{T^{\mu}{}_{\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^{\mu}{}_{\nu} \mathcal{L}}$$

$$\Rightarrow \quad \partial_\mu T^{\mu}{}_{\nu} = 0 \qquad \text{conserved!}$$

For $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$ get $\dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = \partial^\mu \phi$

$$\Rightarrow \quad T^{\mu}{}_{\nu} = \partial^\mu \phi \, \partial_\nu \phi - \delta^{\mu}{}_{\nu} \frac{1}{2} \partial_\rho \phi \, \partial^\rho \phi .$$

$$\Rightarrow \quad T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\rho \phi \, \partial^\rho \phi$$

but: not always symmetric (will see more later)

<u>Conserved charges:</u> for a conserved current j^μ

(such that $\partial_\mu j^\mu = 0$) we have a charge:

$$Q(t) = \int d^3x \; j^0(\vec{x}, t)$$

(e.g. electric charge).

$$\frac{dQ(t)}{dt} = \int d^3x \, \frac{\partial}{\partial t} j^0(\vec{x}, t) = \int d^3x \left[\underbrace{\partial_\mu j^\mu}_{0} - \vec{\nabla} \cdot \vec{j} \right] =$$

$$= - \int d^3x \; \vec{\nabla} \cdot \vec{j} \overset{\text{surface term}}{=} 0$$

For complex scalar field ϕ we had

$$j^\mu = i(\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi) \Rightarrow Q = \int d^3x\, i[\phi\,\partial^0\phi^* - \phi^*\partial^0\phi].$$

For real scalar field we had $T^\mu{}_\nu$ which was conserved: $\partial_\mu T^{\mu\nu} = 0$.

$$\Rightarrow Q^\nu = \int d^3x\, T^{0\nu} \sim 4 \text{ conserved charges } \nu = 0,1,2,3$$

$$\Rightarrow Q^0 = \int d^3x\, T^{00} = \int d^3x \left[\frac{\delta\mathcal{L}}{\delta(\partial_0\phi)}\partial^0\phi - \mathcal{L}\right]$$

In classical mechanics one had a <u>Hamiltonian</u>:

$$H = \sum_i p_i \dot{q}_i - L \Rightarrow \text{we had } p_i = \frac{\delta\mathcal{L}}{\delta\dot{q}^i}.$$

$$\Rightarrow H = \sum_i \frac{\delta\mathcal{L}}{\delta\dot{q}^i}\dot{q}^i - L.$$

The field theory analogue is (remember $L \to \int d^3x\,\mathcal{L}$)

$\dot{\varphi} = \partial_0\varphi$

$$H \equiv \int d^3x \left[\frac{\delta\mathcal{L}}{\delta\dot{\varphi}}\dot{\varphi} - \mathcal{L}\right] \equiv \int d^3x\, \mathcal{H}$$

$\Rightarrow$ we see that

$$Q^0 = \int d^3x \, T^{00} = \int d^3x \, \mathcal{H} = H$$ $\Rightarrow$ this is the Hamiltonian! It is conserved: time translations lead to energy conservation!

$$Q^i = \int d^3x \, T^{0i} = \int d^3x \left[\frac{\delta \mathcal{L}}{\delta \dot{\phi}} \partial^i \phi \right]$$ $\Rightarrow$ interpret as 3-momentum of the field.

$\mu \neq 0$: $\mathcal{H} = (\partial^0 \varphi)^2 - \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m^2}{2}\varphi^2 = \frac{1}{2}(\partial^0\varphi)^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 \geq 0$.

$\Rightarrow \mathcal{H} \geq 0 \Rightarrow$ energy of the field ≥ 0 (not of a particle).

~~Lorentz & Poincaré Groups and Classification of Fields~~

~~Before we start quantizing the fields, let us see what kinds of fields exist.~~

~~This can be accomplished by studying the group of Lorentz transformations.~~

~~We start by reviewing some group theory.~~

Free Dirac Field

in QFT course you will learn that Lorentz group (the group of all Lorentz transformations, that is 3 boosts ⊕ 3 rotations) Λ $x'^\mu = \Lambda^\mu{}_\nu x^\nu$,
$\vec{\beta}$ $\vec{\theta}$
is equivalent to $SU(2) \otimes SU(2)$.

$SU(2)$ is the group of spin (angular momentum).

This group has an ∞ number of representations, corresponding to particles of different spin: $0, \frac{1}{2}, 1, \ldots$

We are interested in spin-1/2 particles: they are represented by Pauli matrices σ^i:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\frac{\sigma^i}{2}$ are the generators of $SU(2)$ with the algebra

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2} \right] = i\, \varepsilon^{ijk} \frac{\sigma^k}{2}.$$

What do σ^i act on? These are 2×2 matrices $\Rightarrow$

$\Rightarrow$ they have to act on 2-component objects

Def. Spinor fields (Weyl spinors)

$$\chi_L = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \qquad \chi_R = \begin{pmatrix} \chi_3 \\ \chi_4 \end{pmatrix}$$

One spinor for each $SU(2)$ group in $SU(2) \times SU(2)$.

Under a general Lorentz $\underbrace{\text{transformation}}^{(x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu)}$ they transform as

$$\left[\begin{array}{l} \chi_L(x) \to \chi_L'(x') = \Lambda_L\, \chi_L(x) \\ \chi_R(x) \to \chi_R'(x') = \Lambda_R\, \chi_R(x) \end{array}\right.$$

where
$$\left.\begin{array}{l} \Lambda_L = e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} + i\vec{\beta})} \\ \Lambda_R = e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} - i\vec{\beta})} \end{array}\right\} \text{representations of } SU(2)$$

$\vec{\theta}$ = rotation angle, direction defines rotation axis, magnitude is the angle

$\vec{\beta} = \frac{\vec{u}}{c} = \vec{u}$ ~ boost parameter, $\vec{u}$ ~ velocity between the frames

$\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ ~ a 3-vector made out of Pauli matrices

χ_L = left-handed spinor, $(\frac{1}{2},0)$ representation

χ_R = right-handed spinor, $(0,\frac{1}{2})$ representation

<u>Parity</u>: $\vec{x} \xrightarrow{\mathbb{P}} -\vec{x} \quad \Rightarrow \quad \vec{\theta} \xrightarrow{\mathbb{P}} \vec{\theta}$ $\quad (x'^i = R^{ij} x^j \Rightarrow$
$\Rightarrow x^i \xrightarrow{\mathbb{P}} -x^i,$

$\vec{\beta} \xrightarrow{\mathbb{P}} -\vec{\beta}$ $\quad R^{ij} \xrightarrow{\mathbb{P}} R^{ij})$

$\Rightarrow \quad \Lambda_L \xrightarrow{\mathbb{P}} \Lambda_R \quad \Rightarrow$ parity relates the two Weyl spinors

$\Rightarrow \chi_L \xrightarrow{\mathbb{P}} \chi_R, \; \chi_R \xrightarrow{\mathbb{P}} \chi_L$

Def. Require that the spinor field is a parity eigenstate (as observed in Nature), need

$(\frac{1}{2},0) \oplus (0,\frac{1}{2}) \Rightarrow$ <u>Dirac spinors</u>

$$\psi_D = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \text{a 4-component object!}$$

$\Rightarrow$ this is the field describing all the fermions in the Standard Model

$\Rightarrow$ we need to construct a Lagrangian describing the dynamics of this field

Classification of fields

$(0,0)$	spin-0	scalar field $\varphi(x)$	1 d.o.f
$(\frac{1}{2},0)$ $(0,\frac{1}{2})$	spin-1/2	left-handed spinor $\chi_L = \binom{\chi_1}{\chi_2}$ right-handed spinor $\chi_R = \binom{\chi_3}{\chi_4}$ Dirac spinor $\Psi : (\frac{1}{2},0)\oplus(0,\frac{1}{2})$	2 d.o.f 2 d.o.f
$(\frac{1}{2},\frac{1}{2})$	spin-1	vector field $A_\mu(x)$ $(\frac{1}{2},\frac{1}{2}) = (\frac{1}{2},0)\otimes(0,\frac{1}{2})$	4 d.o.f.
$(1,0)$	spin-1	$B_{\mu\nu} = -B_{\nu\mu},\ B_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$	3 d.o.f
$(0,1)$	spin-1	$B_{\mu\nu} = -B_{\nu\mu},\ B_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$	3 d.o.f.
$(1,\frac{1}{2})\oplus(\frac{1}{2},1)$	spin-3/2	ψ^μ Rarita-Schwinger field $\gamma^\mu\psi_\mu = 0$ constr.	6+6 d.o.f = = 12 d.o.f. =16-4
$(1,1)$	spin-2	$g_{\mu\nu}$ ~ graviton field $g^\mu{}_\mu = 4$ dim	9 d.o.f. =10-1

$$\left(\text{BTW}, \quad A \oplus B = A \otimes \mathbb{1} + \mathbb{1} \otimes B \right)$$

d.o.f = degrees of freedom = # of independent <u>complex</u> components!

φ ~ can be complex, $\chi_{L,R}$ ~ complex, A_μ can be complex (e.g. W-boson), $B_{\mu\nu}$ is complex, ψ^μ – , – ...

Under Lorentz transformation ψ_D becomes:

$$\psi_D(x) \Rightarrow \psi'_D(x) = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \psi_D(\Lambda^{-1}x).$$

$$= \begin{pmatrix} e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\beta})} & 0 \\ 0 & e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}-i\vec{\beta})} \end{pmatrix}$$

Can we construct a Lorentz-invariant object?

$\psi^+\psi = \chi_L^+\chi_L + \chi_R^+\chi_R$ is not $\mathcal{L}$. inv., as

$\chi_L \to \Lambda_L\chi_L$, $\chi_L^+ \Rightarrow \chi_L^+\Lambda_L^+ = \chi_L^+\Lambda_R^{-1}$

$\Rightarrow \chi_L^+\chi_L = \chi_L^+\Lambda_R^{-1}\Lambda_L\chi_L$ not unitary $\Rightarrow$ not inv.

Def. Dirac γ-matrices (in Weyl representation):

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

$\{A,B\} = AB + BA$

anti-commutator.

$\Rightarrow$ **Def.** $\boxed{\bar{\psi} \equiv \psi^+\gamma^0}$ $\qquad \bar{\psi}_\alpha = (\psi^+)_\beta(\gamma^0)_{\beta\alpha}$

sum, $\alpha, \beta = 1, 2, 3, 4$

$$\bar{\psi} = (\chi_L^+ \; \chi_R^+)\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (\chi_R^+ \; \chi_L^+).$$

$$\bar{\psi}\psi = \psi^+\gamma^0\psi = \begin{pmatrix}\chi_L^+ & \chi_R^+\end{pmatrix}\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}$$

$$= \chi_L^+\chi_R + \chi_R^+\chi_L \quad \Rightarrow \quad \mathcal{L}.\ \text{invariant!}$$

(check: $\chi_L^+\chi_R \to \chi_L^+ \underbrace{\Lambda_L^+}_{\Lambda_R^{-1}} \Lambda_R \chi_R = \chi_L^+\chi_R$!)

Def. $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}$ (Weyl representation)

$$\Rightarrow \bar{\psi}\gamma^5\psi = \begin{pmatrix}\chi_L^+ & \chi_R^+\end{pmatrix}\underbrace{\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}}_{\begin{pmatrix}0 & +1\\ -1 & 0\end{pmatrix}}\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}$$

$$= \chi_L^+\chi_R - \chi_R^+\chi_L \quad \Rightarrow \mathcal{L}.\ \text{inv. too!}$$

but: $\mathbb{P}: \chi_L \to \chi_R, \quad \chi_R \to \chi_L \Rightarrow$

$\Rightarrow \mathbb{P}: \bar{\psi}\gamma^5\psi \to -\bar{\psi}\gamma^5\psi$ changes sign

$\Rightarrow \bar{\psi}\gamma^5\psi \sim$ pseudoscalar.

$$\Rightarrow \boxed{\begin{array}{l}\bar{\psi}\psi \sim \text{Lorentz scalar}\\[1ex] \bar{\psi}\gamma^5\psi \sim \text{pseudoscalar}\end{array}}$$

But: we need to find a Lagrangian $\Rightarrow$ need ∂_μ's

$\Rightarrow$ need vectors!