Last time | We derived the energy-momentum tensor expression

$$\boxed{T^{\mu}{}_{\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \partial_\nu \varphi - \delta^{\mu}{}_{\nu} \mathcal{L}}$$

It is conserved, $\partial_\mu T^{\mu}{}_{\nu} = 0$.

$T^{00} \sim$ energy density, $T^{0i} \sim$ momentum density

Free Dirac Field

(Def.) Weyl spinors:

$$\chi_L = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \chi_R = \begin{pmatrix} \chi_3 \\ \chi_4 \end{pmatrix}$$

left-handed right-handed

$$\begin{cases} \chi_L(x) \to \chi_L'(x') = \Lambda_L \chi_L(x) \\ \chi_R(x) \to \chi_R'(x') = \Lambda_R \chi_R(x) \end{cases}$$

Lorentz transformation for Weyl spinors

$$\Lambda_{L,R} = e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} \pm i\vec{\beta})}$$

(Def.) Dirac spinor (parity eigenstate)

$$\psi_D = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$\psi_D(x) \rightarrow \psi_D'(x') = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \psi_D(x)$$

Lorentz transform for Dirac spinors

Under Lorentz transformation ψ_D becomes:

$$\psi_D(x) \Rightarrow \psi'_D(x) = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \psi_D(\Lambda^{-1}x).$$

$$\parallel$$

$$\begin{pmatrix} e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\beta})} & 0 \\ 0 & e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}-i\vec{\beta})} \end{pmatrix}$$

Can we construct a Lorentz-invariant object?

$\psi^\dagger\psi = \chi_L^\dagger\chi_L + \chi_R^\dagger\chi_R$ is not $\mathcal{L}$. inv., as

$\chi_L \to \Lambda_L\chi_L$, $\chi_L^\dagger \Rightarrow \chi_L^\dagger\Lambda_L^\dagger = \chi_L^\dagger\Lambda_R^{-1}$

$\Rightarrow \chi_L^\dagger\chi_L = \chi_L^\dagger\Lambda_R^{-1}\Lambda_L\chi_L$ not unitary $\Rightarrow$ not inv.

Def. Dirac γ-matrices (in Weyl representation):

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

$\{A,B\} = AB + BA$ anti-commutator.

$\Rightarrow$ Def. $\boxed{\bar{\psi} \equiv \psi^\dagger\gamma^0}$ $\quad \bar{\psi}_\alpha = (\psi^\dagger)_\beta(\gamma^0)_{\beta\alpha}$

sum, $\alpha,\beta = 1,2,3,4$

$$\bar{\psi} = (\chi_L^\dagger\ \chi_R^\dagger)\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (\chi_R^\dagger\ \chi_L^\dagger).$$

$$\bar\Psi\psi = \psi^+\gamma^0\psi = \begin{pmatrix}\chi_L^+ & \chi_R^+\end{pmatrix}\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}$$

$$= \chi_L^+\chi_R + \chi_R^+\chi_L \quad \Rightarrow \quad \mathcal{L}\text{. invariant!}$$

(check: $\chi_L^+\chi_R \to \chi_L^+ \underbrace{\Lambda_L^+}_{\Lambda_R^{-1}} \Lambda_R \chi_R = \chi_L^+\chi_R$!)

Def. $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}$ (Weyl representation)

$$\Rightarrow \bar\Psi\gamma^5\psi = \begin{pmatrix}\chi_L^+ & \chi_R^+\end{pmatrix}\underbrace{\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}}_{\begin{pmatrix}0 & +1\\ -1 & 0\end{pmatrix}}\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}$$

$$= \chi_L^+\chi_R - \chi_R^+\chi_L \quad \Rightarrow \mathcal{L}\text{. inv. too!}$$

but: $\mathbb{P}: \chi_L \to \chi_R, \quad \chi_R \to \chi_L \Rightarrow$

$\Rightarrow \mathbb{P}: \bar\Psi\gamma^5\psi \to -\bar\Psi\gamma^5\psi$ changes sign

$\Rightarrow \bar\Psi\gamma^5\psi \sim$ pseudoscalar.

$$\Rightarrow \boxed{\begin{array}{l}\bar\Psi\psi \sim \text{Lorentz scalar}\\[1em] \bar\Psi\gamma^5\psi \sim \text{pseudoscalar}\end{array}}$$

But: we need to find a Lagrangian $\Rightarrow$ need ∂_μ's

$\Rightarrow$ need vectors!

What is a 4-vector field? How does it transform under Lorentz transformations?

If $\varphi(x)$ a scalar field $\Rightarrow$ $\partial_\mu \varphi(x) = A_\mu(x)$ is a 4-vector field.

$$\varphi(x) \to \varphi'(x') = \varphi(x)$$

$$\partial_\mu \varphi(x) \to \partial'_\mu \varphi'(x') = \partial'_\mu \varphi(x) = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \varphi(x).$$

Now, $x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow x' = \Lambda \cdot x \Rightarrow x = \Lambda^{-1} \cdot x'$

$$\Rightarrow x^\nu = (\Lambda^{-1})^\nu{}_\mu x'^\mu \Rightarrow \frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu .$$

As $\eta^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta} \Rightarrow \delta^\mu{}_\nu = \Lambda^\mu{}_\alpha \Lambda_{\nu\beta} \eta^{\alpha\beta}$

$$= \Lambda^\mu{}_\alpha \Lambda_\nu{}^\alpha = \Lambda^\mu{}_\alpha \cdot (\Lambda^{-1})^\alpha{}_\nu \Rightarrow$$

$$(\Lambda^{-1})^\alpha{}_\nu = \Lambda_\nu{}^\alpha$$

Thus $\dfrac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu = \Lambda_\mu{}^\nu$

$$\Rightarrow \partial'_\mu \varphi'(x') = A'_\mu = \Lambda_\mu{}^\nu A_\nu \Rightarrow$$

$$\boxed{\begin{aligned} A_\mu &\to A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x) \\ A^\mu &\to A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x) \end{aligned}}$$ as expected!

Combine Dirac matrices into $\gamma^\mu = (\gamma^0, \vec{\gamma})$

$\mu = 0, 1, 2, 3$.

$\Rightarrow$ consider $\bar{\psi}\gamma^\mu\psi$. Claim: it's a 4-vector!

Check: rotations

$$\psi \to \psi' = \begin{pmatrix} e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}} & 0 \\ 0 & e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}} \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

$\Rightarrow \bar{\psi}\gamma^\mu\psi \Rightarrow$ 0th component is

$\bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi \Rightarrow$ invariant under rotations.

spatial component:

$$\bar{\psi}\gamma^i\psi = \psi^\dagger\gamma^0\gamma^i\psi = \psi^\dagger\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\psi =$$

$$= \psi^\dagger\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\psi = (\chi_L^\dagger \; \chi_R^\dagger)\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} =$$

$$= -\chi_L^\dagger\sigma^i\chi_L + \chi_R^\dagger\sigma^i\chi_R \longrightarrow -\chi_L^\dagger e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\sigma^i e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi_L$$

$$+ \chi_R^\dagger e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\sigma^i e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi_R \simeq (\text{infinitesimal}) =$$

$$= -\chi_L^\dagger\underbrace{\left(1 + \frac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right)\sigma^i\left(1 - \frac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right)}\chi_L + (L \to R)$$

$$= \sigma^i + \frac{i}{2}\theta^j\underbrace{[\sigma^j, \sigma^i]}_{2i\varepsilon^{jik}\sigma^k} + \ldots = \sigma^i + \varepsilon^{ijk}\theta^j\sigma^k$$

$\Rightarrow \bar{\psi}\gamma^i\psi \to \bar{\psi}\gamma^i\psi + \varepsilon^{ijk}\theta^j\bar{\psi}\gamma^k\psi$ ~ just like a rotation! (clockwise by angle θ)

One can show that the object $\bar{\psi}\gamma^\mu\psi$ transforms under boosts as expected of a 4-vector too $\Rightarrow$

$\bar{\psi}\gamma^\mu\psi$ is a 4-vector!

$\mathbb{P}$: $\bar{\psi}\gamma^0\psi = \psi^+\psi = \chi_L^+\chi_L + \chi_R^+\chi_R$ ~ invariant

$$\bar{\psi}\gamma^i\psi = -\chi_L^+\sigma^i\chi_L + \chi_R^+\sigma^i\chi_R$$

$$\Rightarrow \quad \bar{\psi}\gamma^i\psi \xrightarrow{\mathbb{P}} -\bar{\psi}\gamma^i\psi \Rightarrow \text{polar vector!}$$

$\bar{\psi}\gamma^\mu\gamma^5\psi$ is a pseudo-vector (axial vector)

In general can show that

$$\psi \to \psi'_{(x')} = e^{-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}\psi(x)$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ ~ a reducible representation of Lorentz algebra ; or $\psi' = e^{-\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}}\psi$, $S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$.

Lagrangian for Dirac spinors:

$$\mathcal{L} = A\,\bar{\psi}\gamma^\mu\partial_\mu\psi + B\,\bar{\psi}\psi$$

L. invariants.

EOM: $$\frac{\delta\mathcal{L}}{\delta\bar{\psi}} - \partial_\mu\underbrace{\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\right]}_{=0} = 0 \Rightarrow (A\gamma^\mu\partial_\mu + B)\psi = 0$$

$\Rightarrow$ act with $\gamma^\nu \partial_\nu$

$$\Rightarrow \left[A \underbrace{\gamma^\nu \gamma^\mu}_{} \partial_\nu \partial_\mu + B \gamma^\nu \partial_\nu\right] \psi = 0$$

$$\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$$

$$\left[A\, \partial^2 + B \gamma^\nu \partial_\nu\right] \psi = 0$$

Now, $\gamma^\mu \partial_\mu \psi = -\frac{B}{A}\psi \Rightarrow \left[A\, \partial^2 - \frac{B^2}{A}\right]\psi = 0$

$$\Rightarrow \left[\partial^2 - \frac{B^2}{A^2}\right]\psi = 0$$

c.f. Klein-Gordon eqn: $(\partial^2 + m^2)\phi = 0 \Rightarrow$ gives $p^2 = m^2$ ~ correct on-shell condition

$\Rightarrow \frac{B^2}{A^2} = -m^2 \Rightarrow$ pick $A = i,\ B = -m$

$$\Rightarrow \boxed{\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi}$$

free Dirac field Lagrangian

EOM: $\boxed{\left[i\gamma^\mu \partial_\mu - m\right]\psi(x) = 0}$ Dirac equation.

Why no $\bar{\psi}\partial_\mu \partial^\mu \psi$ term in $\mathcal{L}$? For instance, dimensions:

$\dim \psi = 3/2 \Rightarrow \dim[\bar{\psi}\partial^2\psi] = M^5 \Rightarrow$ need dimensionful coupling

$\Rightarrow$ not free field anymore.

(Ultimately experimental fact.)

$\bar{\psi}\Box\psi$ is like 4 scalar fields, one'd need to impose L. transf rules by hand to get $\chi_{L,R} \to \Lambda_{L,R}\chi_{L,R}$

$$\mathcal{L}_{\text{Dirac}} = A\bar{\psi}\not{\partial}\psi + B\bar{\psi}\psi$$

EOM: $$\frac{\delta\mathcal{L}}{\delta\psi} - \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\right] = 0$$

$$B\bar{\psi} - \partial_\mu\left[A\bar{\psi}\gamma^\mu\right] = 0$$

EOM: $$\frac{\delta\mathcal{L}}{\delta\bar{\psi}} - \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\right] = 0$$

$$A\not{\partial}\psi + B\psi = 0$$

$$\Downarrow$$

$$B^*\bar{\psi} + A^*(\partial_\mu\bar{\psi})\gamma^\mu = 0$$

$\Rightarrow$ $B \sim B^*$, $A \sim -A^*$

$\Rightarrow$ B ~ real, A ~ imaginary.

(also needed for <u>hermiticity</u>).

Dirac Lagrangian is

$$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m\right]\psi$$

Any symmetries? Yes, we have $\mathcal{L} \to \mathcal{L}$ under

$\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to e^{-i\alpha}\bar{\psi}$, α a real number

$\Rightarrow \delta\mathcal{L} = 0 \Rightarrow$ remember we had for scalar fields

$$\delta\mathcal{L} = \sum_a \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu \varphi^a)}\delta\varphi^a\right]$$

$\Rightarrow$ similarly for spinors ψ & $\bar{\psi}$ we have

$$\delta\mathcal{L} = \partial_\mu\left[\underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}}_{= i\bar{\psi}\gamma^\mu}\delta\psi + \underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}}_{=0}\delta\bar{\psi}\right] = 0$$

(as $\mathcal{L}$ is invariant under $U(1)$),

as nothing in $\mathcal{L}$ depends on $\partial_\mu\bar{\psi}$

$\Rightarrow$ get $\partial_\mu\left[i\bar{\psi}\gamma^\mu\delta\psi\right] = 0$

$\delta\psi = (1 + i\alpha + \ldots)\psi - \psi = i\alpha\psi \Rightarrow$ get

$$\partial_\mu\left[\bar{\psi}\gamma^\mu\psi\right] = 0 \Rightarrow \boxed{j^\mu = \bar{\psi}\gamma^\mu\psi}$$

EM current.

is a conserved current: $\partial_\mu j^\mu = 0$. (can check explicitly)

In general can construct any bilinear object $\bar{\psi}\Gamma\psi$, with Γ a 4×4 matrix. "Full basis" of 4×4 matrices with definite Lorentz-transform properties is

$$\boxed{\Gamma = \{ \mathbb{1}, \gamma^\mu, \gamma^5, \gamma^\mu\gamma^5, \sigma^{\mu\nu} \}}$$

Demand $\bar{\psi}\Gamma\psi$ to be hermitean $\Rightarrow$ $\gamma^0\Gamma^\dagger\gamma^0 = \Gamma$ to make herm. 16 conditions $\Rightarrow$ $32-16=16$ d.o.f. replace $\gamma^5 \to i\gamma^5$, $\gamma^5\gamma^\mu \to i\gamma^5\gamma^\mu$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. 16 matrices.

One has:

bilinear	transformation law
$\bar{\psi}\psi$	scalar
$\bar{\psi}\gamma^5\psi$	pseudoscalar
$\bar{\psi}\gamma^\mu\psi$	vector
$\bar{\psi}\gamma^\mu\gamma^5\psi$	axial vector
$\bar{\psi}\sigma^{\mu\nu}\psi$	antisymmetric tensor

$j_\mu^5 = \bar{\psi}\gamma^5\gamma_\mu\psi$ is also a 4-vector (axial current)

Is it conserved? In fact $\partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\psi$

$\Rightarrow$ it is conserved only if $m=0$.

<u>Energy-momentum tensor</u>: $T_{\mu\nu} = \dfrac{\delta\mathcal{L}}{\delta(\partial^\mu\psi)}\partial_\nu\psi + \cancelto{0}{\dfrac{\delta\mathcal{L}}{\delta(\partial^\mu\bar{\psi})}}\partial_\nu\bar{\psi} - g_{\mu\nu}\mathcal{L}$ by analogy with scalar field.

Any 4×4 matrix $M_{4\times4} = \sum_{i=1}^{16} c_i\Gamma_i$ with <u>complex</u> coefficients c_i $\Rightarrow$ only 16 matrices Γ_i.

We get $T_{\mu\nu} = i\bar{\Psi}\gamma_\mu \partial_\nu \Psi - g_{\mu\nu}\left[\bar{\Psi}(i\gamma^\alpha\partial_\alpha - m)\Psi\right]$

$$\Rightarrow \quad T_{\mu\nu} = \bar{\Psi}\left[i\gamma_\mu\partial_\nu - g_{\mu\nu}\, i\gamma^\alpha\partial_\alpha + g_{\mu\nu} m\right]\Psi$$

However, we can simplify this by using Dirac equation $(i\gamma^\alpha\partial_\alpha - m)\Psi = 0 \Rightarrow$ get

$$T_{\mu\nu} = i\bar{\Psi}\gamma_\mu\partial_\nu\Psi \qquad \text{(not symmetric)}$$

Remember that the Hamiltonian $H = \int d^3x\, T_{00}$.

We get $H = \int d^3x\; i \underbrace{\bar{\Psi}\gamma^0}_{\Psi^+ \underbrace{\gamma^0\gamma^0}_{\mathbb{1}}} \partial_t \Psi = \int d^3x\; i\Psi^+\partial_t\Psi$

$$\Rightarrow \quad H = \int d^3x\; i\Psi^+\partial_t\Psi \qquad \text{problem: } H \text{ is not } \geq 0!$$

(This is different from scalar fields, for which H was ≥ 0 for the field!)

$T_{\mu\nu}$ can be symmetrized: $T^{\text{symm.}}_{\mu\nu} = i\bar{\Psi}\left[\frac{1}{2}\left(\gamma_\mu\overleftrightarrow{\partial}_\nu + \gamma_\nu\overleftrightarrow{\partial}_\mu\right)\right]\Psi$ $\quad \frac{1}{4}$

~ can show that $\partial^\mu T^{\text{symm}}_{\mu\nu} = 0$.

Here $\bar{\Psi}\overleftrightarrow{\partial}_\mu\Psi = \bar{\Psi}\partial_\mu\Psi - (\partial_\mu\bar{\Psi})\Psi$.

<u>Useful γ-matrix formulas:</u>

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \qquad (\gamma^0)^\dagger = \gamma^0, \ (\gamma^i)^\dagger = -\gamma^i$$

$$\text{''} \ \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad , \quad (\gamma^5)^\dagger = \gamma^5$$

$$(\gamma^0)^2 = -(\gamma^i)^2 = \mathbb{1}, \quad (\gamma^5)^2 = \mathbb{1}.$$

easy to see that $(\gamma^\mu)^2 = g^{\mu\mu}$ (no summation)

$$\{\gamma^5, \gamma^\mu\} = 0$$

(Easy to check.)

Also $\gamma_\mu\gamma^\mu = 4$, $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$.

Finally, note that $\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$ in Weyl basis

$\Rightarrow$ Def. $\boxed{P_L = \dfrac{\mathbb{1} - \gamma^5}{2}, \quad P_R = \dfrac{\mathbb{1} + \gamma^5}{2}}$

$\Rightarrow$ $P_L \psi = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} \equiv \psi_L$ projection on left-handed spinor

$P_R \psi = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} \equiv \psi_R$ —"— right-handed

Can check that $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L P_R = P_R P_L = 0$.

($\psi_L \sim$ helicity $-1/2$, $\psi_R \sim$ helicity $+1/2$, more later)