## Last time | Free Dirac Field (cont'd)

We derived the Lagrangian for Dirac spinor field:

$$\boxed{\mathcal{L}_{Dirac} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi}$$

where $\not{\partial} \equiv \gamma^\mu \partial_\mu$

$$\bar{\psi} = \psi^\dagger \gamma^0.$$

EOM gave us Dirac equation:

$$\boxed{[i\not{\partial} - m]\psi(x) = 0}$$

$\mathcal{L}_{Dirac}$ is invariant under $\psi(x) \to e^{i\alpha}\psi(x)$

$\alpha$ = real #

$\Rightarrow$ $\boxed{\partial_\mu j^\mu = 0}$ , $j^\mu = \bar{\psi}\gamma^\mu\psi$ EM current (vector current)

Energy-momentum tensor:

$$\boxed{T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi.}$$

Hamiltonian: $H = \int d^3x \, i\psi^\dagger\partial_t\psi$ , not $\geq 0$.

Dirac Lagrangian is

$$\mathcal{L} = \bar{\psi}\left[ i\gamma^\mu \partial_\mu - m \right]\psi$$

Any symmetries? Yes, we have $\mathcal{L} \to \mathcal{L}$ under

$\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to e^{-i\alpha}\bar{\psi}$, $\alpha$ a real number

$\Rightarrow \delta\mathcal{L} = 0 \Rightarrow$ remember we had for scalar fields

$$\delta\mathcal{L} = \sum_a \partial_\mu \left[ \frac{\delta\mathcal{L}}{\delta(\partial_\mu \varphi^a)} \delta\varphi^a \right]$$

$\Rightarrow$ similarly for spinors $\psi$ & $\bar{\psi}$ we have

$$\delta\mathcal{L} = \partial_\mu \left[ \underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_\mu \psi)}}_{= i\bar{\psi}\gamma^\mu} \delta\psi + \underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_\mu \bar{\psi})}}_{= 0} \delta\bar{\psi} \right] = 0$$

(as $\mathcal{L}$ is invariant under $U(1)$),

as nothing in $\mathcal{L}$ depends on $\partial_\mu \bar{\psi}$

$\Rightarrow$ get $\partial_\mu \left[ i\bar{\psi}\gamma^\mu \delta\psi \right] = 0$

$\delta\psi = (1 + i\alpha + \ldots)\psi - \psi = i\alpha\psi \Rightarrow$ get

$$\partial_\mu \left[ \bar{\psi}\gamma^\mu \psi \right] = 0 \Rightarrow \boxed{j^\mu = \bar{\psi}\gamma^\mu\psi}$$ EM current.

is a conserved current: $\partial_\mu j^\mu = 0$ (can check explicitly)

In general can construct any bilinear object $\bar{\psi}\Gamma\psi$, with $\Gamma$ a $4\times4$ matrix. "Full basis" of $4\times4$ matrices with definite Lorentz-transform properties is

$$\boxed{\Gamma = \{ \mathbb{1}, \gamma^\mu, \gamma^5, \gamma^\mu\gamma^5, \sigma^{\mu\nu} \}}$$

Demand $\bar{\psi}\Gamma\psi$ to be hermitean $\Rightarrow$ $\gamma^0\Gamma^\dagger\gamma^0 = \Gamma$ 16 conditions $\Rightarrow$ $\Rightarrow 32-16 = 16$ d.o.f.

to make herm. replace $\begin{cases} \gamma^5 \to i\gamma^5 \\ \gamma^5\gamma^\mu \to i\gamma^5\gamma^\mu \end{cases}$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. 16 matrices.

One has:

| bilinear | transformation law |
|---|---|
| $\bar{\psi}\psi$ | scalar |
| $\bar{\psi}\gamma^5\psi$ | pseudoscalar |
| $\bar{\psi}\gamma^\mu\psi$ | vector |
| $\bar{\psi}\gamma^\mu\gamma^5\psi$ | axial vector |
| $\bar{\psi}\sigma^{\mu\nu}\psi$ | antisymmetric tensor |

$j_\mu^5 = \bar{\psi}\gamma^5\gamma_\mu\psi$ is also a 4-vector (axial current)

Is it conserved? In fact $\partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\psi$

$\Rightarrow$ it is conserved only if $m=0$.

<u>Energy-momentum tensor</u>: $T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial^\mu\psi)}\partial_\nu\psi + \cancel{\frac{\delta\mathcal{L}}{\delta(\partial^\mu\bar{\psi})}}^{\,0}\partial_\nu\bar{\psi} - g_{\mu\nu}\mathcal{L}$ by analogy with scalar field.

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Any $4\times4$ matrix $M_{4\times4} = \sum_{i=1}^{16} c_i\Gamma_i$ with <u>complex</u> coefficients $c_i$ $\Rightarrow$ only 16 matrices $\Gamma_i$.

We get $T_{\mu\nu} = i\bar{\psi}\gamma_\mu \partial_\nu \psi - g_{\mu\nu}\left[\bar{\psi}(i\gamma^\alpha\partial_\alpha - m)\psi\right]$

$$\Rightarrow \quad T_{\mu\nu} = \bar{\psi}\left[i\gamma_\mu\partial_\nu - g_{\mu\nu}\, i\gamma^\alpha\partial_\alpha + g_{\mu\nu} m\right]\psi$$

However, we can simplify this by using Dirac equation $(i\gamma^\alpha\partial_\alpha - m)\psi = 0 \Rightarrow$ get

$$T_{\mu\nu} = i\bar{\psi}\gamma_\mu\partial_\nu\psi \qquad (\text{not symmetric})$$

Remember that the Hamiltonian $H = \int d^3x\, T_{00}$.

We get $H = \int d^3x\; i\underbrace{\bar{\psi}}_{\psi^+\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}}\gamma^0\partial_t\psi = \int d^3x\; i\psi^+\partial_t\psi$

$$\Rightarrow \quad H = \int d^3x\; i\psi^+\partial_t\psi \qquad \text{problem: } H \text{ is not } \geq 0!$$

(This is different from scalar fields, for which $H$ was $\geq 0$ for the field!)

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$T_{\mu\nu}$ can be symmetrized: $T^{\text{symm.}}_{\mu\nu} = i\bar{\psi}\left[\tfrac{1}{2}\left(\gamma_\mu\overleftrightarrow{\partial}_\nu + \gamma_\nu\overleftrightarrow{\partial}_\mu\right)\right]\psi$ $\frac{1}{4}$

$\rightarrow$ can show that $\partial^\mu T^{\text{symm}}_{\mu\nu} = 0$.

Here $\bar{\psi}\overleftrightarrow{\partial}_\mu\psi = \bar{\psi}\partial_\mu\psi - (\partial_\mu\bar{\psi})\psi$.

Useful $\gamma$-matrix formulas:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad (\gamma^0)^+ = \gamma^0, \; (\gamma^i)^+ = -\gamma^i$$

" $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad , \quad (\gamma^5)^+ = \gamma^5$$

$$(\gamma^0)^2 = -(\gamma^i)^2 = \mathbb{1}, \quad (\gamma^5)^2 = \mathbb{1}.$$

easy to see that $(\gamma^\mu)^2 = g^{\mu\mu}$ (no summation)

$$\{\gamma^5, \gamma^\mu\} = 0$$

(Easy to check.)

Also $\gamma_\mu\gamma^\mu = 4$ , $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$.

Finally, note that $\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$ in Weyl basis

$\Rightarrow$ Def. $\boxed{P_L = \frac{\mathbb{1} - \gamma^5}{2}, \quad P_R = \frac{\mathbb{1} + \gamma^5}{2}}$

$\Rightarrow$ $P_L \psi = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} \equiv \psi_L$ projection on left-handed spinor

$P_R \psi = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} \equiv \psi_R$ —"— right-handed

Can check that $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L P_R = P_R P_L = 0$.

($\psi_L \sim$ helicity $-1/2$, $\psi_R \sim$ helicity $+1/2$, more later)

Take Dirac equation $\left[i\gamma^\mu\partial_\mu - m\right]\psi(x) = 0$.

In momentum space $\psi(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\,\psi(p)$

$\Rightarrow \left(\gamma^\mu p_\mu - m\right)\psi(p) = 0$; If $m=0 \Rightarrow \gamma^\mu p_\mu \psi(p) = 0$

as $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ & $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \Rightarrow$

$\Rightarrow \underbrace{\gamma^\mu p_\mu}_{\gamma^0 p_0 + \gamma^i p_i} = \begin{pmatrix} 0 & p_0 - \vec{p}\cdot\vec{\sigma} \\ p_0 + \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix} \Rightarrow$ Dirac equation becomes:

$$\begin{pmatrix} 0 & p_0 - \vec{p}\cdot\vec{\sigma} \\ p_0 + \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = 0$$

Def.

Helicity operator $h \equiv \frac{\vec{p}\cdot\vec{S}}{|\vec{p}|}$ $\Rightarrow$ for spin-$\frac{1}{2}$ particles have $\vec{S} = \frac{1}{2}\vec{\sigma} \Rightarrow h = \frac{1}{2|\vec{p}|}\,\vec{p}\cdot\vec{\sigma}$.

Physical meaning: projection of spin on $\vec{p}$ direction.

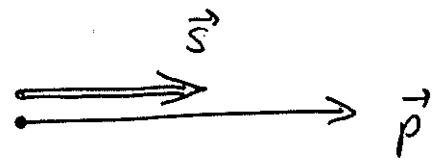
We get
$$(p_0 - \vec{p}\cdot\vec{\sigma})\,\chi_R = 0$$
$$(p_0 + \vec{p}\cdot\vec{\sigma})\,\chi_L = 0$$

Hence, as $|\vec{p}| = p^0$ we get
$$h\,\chi_R = +\tfrac{1}{2}\chi_R$$
$$h\,\chi_L = -\tfrac{1}{2}\chi_L$$

$\Rightarrow$ $\chi_R$ has helicity $+1/2$

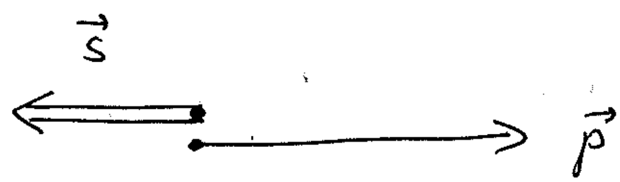
$\chi_L$ — " — $-1/2$

$\Rightarrow$ $\chi_R$ is called right-handed as




the spin is $\Uparrow$ to $\vec{p}$. (hence $h = +1/2$)

$\Rightarrow$ $\chi_L$ is called left-handed as




the spin is $\downarrow\uparrow$ to $\vec{p}$ (hence $h = -1/2$).

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__chirality:__ boost-invariant, but does not commute with the Dirac Hamiltonian for $m \neq 0$.

__helicity:__ not boost-invariant for $m \neq 0$, but always commutes with $H_{Dirac}$ $\Rightarrow$ conserved.

# Vector Field $A_\mu$

Consider a vector field $A_\mu(x)$. What is its Lagrangian? ( $A_\mu(x) \to A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x)$ ~ Lorentz transform )

Start with Dirac field $\psi(x)$ with Lagrangian

$$\mathcal{L} = \bar\psi\left[i\gamma^\mu \partial_\mu - m\right]\psi.$$

This Lagrangian is invariant under

$$\begin{cases} \psi(x) \to e^{i\alpha}\psi(x) \\ \bar\psi(x) \to e^{-i\alpha}\bar\psi(x). \end{cases} \qquad \text{(global symmetry)}$$

<u>Global symmetry</u>: $\alpha$ is independent of $x^\mu$, the same transformation for all points in space.

Let's make it a <u>local symmetry</u>: we want the Lagrangian to be symmetric under

$$\begin{cases} \psi(x) \to e^{i\alpha(x)}\psi(x) \\ \bar\psi(x) \to e^{-i\alpha(x)}\bar\psi(x) \end{cases} \qquad \text{(local symmetry)}$$

where $\alpha(x)$ is a (real-valued) function now.

What happens to $\mathcal{L}$?

$$\bar\psi\left[i\gamma^\mu\partial_\mu - m\right]\psi \to \bar\psi e^{-i\alpha(x)}\left[i\gamma^\mu\partial_\mu - m\right]e^{i\alpha(x)}\psi =$$

$$= \bar{\psi}\,[i\gamma^\mu \partial_\mu - m]\,\psi \;-\; \bar{\psi}\,\gamma^\mu (\partial_\mu \alpha)\,\psi$$

$\Rightarrow$ $\mathcal{L}_{Dirac}$ is not invariant under local $U(1)$ sy(!

$\Rightarrow$ Fix this by introducing gauge field $A_\mu$:

$$\mathcal{L} = \bar{\psi}\,[\,i\gamma^\mu D_\mu - m\,]\,\psi$$

where

Def. $D_\mu = \partial_\mu + ie A_\mu$ is the <u>covariant derivative</u>.

$$\mathcal{L}_1 = \bar{\psi}\,[i\gamma^\mu\partial_\mu - m]\,\psi - e\bar{\psi}\gamma^\mu\psi A_\mu$$

(Remember in E&M: $\mathcal{L}_{int.} = -j^\mu A_\mu$ ~ interaction Lagrangian

$\Rightarrow$ in Dirac theory conserved $U(1)$ current

is $j^\mu = e\bar{\psi}\gamma^\mu\psi$ $\Rightarrow$ $\mathcal{L}_{int} = -j^\mu A_\mu = -e\bar{\psi}\gamma^\mu\psi A_\mu$.

~ we have not done anything new compared to E&M.)

Demand that the $U(1)$ local transformation is

$$\begin{cases} \psi(x) \to e^{i\alpha(x)}\,\psi(x)\,, \quad \bar{\psi} \to e^{-i\alpha}\,\bar{\psi}. \\ A_\mu(x) \to A_\mu - \frac{1}{e}\,\partial_\mu\alpha\,. \end{cases}$$

(Then

$$\mathcal{L}_1 \to \bar{\psi}\,[i\gamma^\mu\partial_\mu - m]\,\psi - \cancel{\bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi} - e\bar{\psi}\gamma^\mu\psi \cdot \left(A_\mu - \cancel{\tfrac{1}{e}\partial_\mu\alpha}\right) = \mathcal{L}_1$$

$\Rightarrow$ now $\mathcal{L}_1$ is invariant!

However, we need a Lagrangian for $A_\mu$ field itself! We impose usual requirements (like for scalar field $\varphi$): at most quadratic in $A_\mu$, $\partial_\mu$. On top of that require that $\mathcal{L}_{A_\mu}$ is gauge-invariant, i.e., invariant under

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha .$$

Start by constucting a gauge-invariant field strength tensor $F_{\mu\nu}$:

Def. $$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$ (field strength tensor)

There is also dual field strength tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$\varepsilon^{0123} = +1 , \quad \varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\nu\mu\rho\sigma} , \ldots$$

Lorentz-invariants are:

$$F_{\mu\nu}F^{\mu\nu} , \quad F_{\mu\nu}\tilde{F}^{\mu\nu} , \quad \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$$

However, $F_{\mu\nu}F^{\mu\nu} = -\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \Rightarrow$ down to two.

Now, one can easily show that $F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu K^\mu$

with some 4-vector $K^\mu$. Hence

$$\int d^4x \; F_{\mu\nu} \tilde{F}^{\mu\nu} = \int d^4x \; \partial_\mu K^\mu = \int_{\substack{\text{surface} \\ \text{(3dim)}}} d\sigma_\mu \; K^\mu \quad \Rightarrow$$

$\Rightarrow$ this is the surface term, does not give any classical dynamics $\Rightarrow$ no good to use $F_{\mu\nu}\tilde{F}^{\mu\nu}$ for Lagrangian.

$\Rightarrow$ only $F_{\mu\nu} F^{\mu\nu}$ is left.

Matching the constants on E&M get

$$\mathcal{L}_{gauge} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

The full Lagrangian for spinors (electrons) interacting with gauge field (photons) is

$$\mathcal{L}_{QED} = \bar{\psi} \left[ i \gamma^\mu D_\mu - m \right] \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

QED = Quantum Electro Dynamics.

$$\mathcal{L}_{QED} = \underbrace{\underbrace{\bar{\psi} \left[ i\gamma^\mu \partial_\mu - m \right] \psi}_{\text{Dirac}} - \underbrace{\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{gauge}}}_{\text{free field terms}} \underbrace{- e \bar{\psi} \gamma^\mu \psi A_\mu}_{\substack{\text{interaction term} \\ \text{(consider later)}}}$$

Take $\boxed{\mathcal{L}_{E\&M} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu}$ .

Let's find its equations of motion. Euler-Lagrange equations are:

$$\frac{\delta \mathcal{L}}{\delta A_\mu} - \partial_\nu \left[ \frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} \right] = 0.$$

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = -j^\mu ; \quad \frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} = -\frac{1}{4} \frac{\delta(F_{\alpha\beta} F^{\alpha\beta})}{\delta(\partial_\nu A_\mu)} =$$

$$= -\frac{1}{4} g^{\alpha\rho} g^{\beta\sigma} \frac{\delta(F_{\alpha\beta} F_{\rho\sigma})}{\delta(\partial_\nu A_\mu)} = -\frac{1}{4} g^{\alpha\rho} g^{\beta\sigma} \Big[ (\delta_\alpha^\nu \delta_\beta^\mu -$$

$$- \delta_\alpha^\mu \delta_\beta^\nu) F_{\rho\sigma} + F_{\alpha\beta} (\delta_\rho^\nu \delta_\sigma^\mu - \delta_\rho^\mu \delta_\sigma^\nu) \Big] =$$

$$= -\frac{1}{4} \left[ F^{\nu\mu} - F^{\mu\nu} + F^{\nu\mu} - F^{\mu\nu} \right] = F^{\mu\nu}.$$

$$\Rightarrow \frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} = F^{\mu\nu}$$

$\Rightarrow$ EOM are $-j^\mu - \partial_\nu F^{\mu\nu} = 0$

$\Rightarrow \boxed{\partial_\nu F^{\nu\mu} = j^\mu}$ Maxwell equations (as expected)

$\boxed{F_{\mu\nu} = [D_\mu D_\nu - D_\nu D_\mu] \frac{-i}{e}}$ (useful formula)

$$\partial_\nu F^{\nu\mu} = j^\mu \Rightarrow \partial_\nu (\partial^\nu A^\mu - \partial^\mu A^\nu) = j^\mu$$

$$\Rightarrow \boxed{\Box A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu} \Rightarrow \text{in Lorenz gauge, } \partial_\mu A^\mu = 0$$

$$\Rightarrow \boxed{\Box A^\mu = j^\mu}$$

in E & M we had

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}.$$