Last time

Vector Field A_μ

$$\mathcal{L}_{QED} = \bar{\psi}[i\not{D} - m]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$= \bar{\psi}[i\not{\partial} - m]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\,\bar{\psi}\gamma^\mu\psi A_\mu$$

$D_\mu = \partial_\mu + ieA_\mu$ ~ covariant derivative

$F_{\mu\nu} = -\frac{i}{e}[D_\mu, D_\nu]$ ~ field strength tensor

$\mathcal{L}_{QED}$ was obtained by starting with the Dirac Lagrangian and requiring that local gauge symmetry be true for the new Lagrangian:

$$\begin{cases} \psi(x) \to e^{i\alpha(x)}\psi(x) \\ \bar{\psi}(x) \to e^{-i\alpha(x)}\psi(x) \\ A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \end{cases}$$

Canonical Quantization

Here we will quantize free fields:

scalar, spinor & vector fields.

We'll start with a real scalar field $\varphi(x)$.

Very Brief Review

Scalar field $\varphi(x)$: $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2$

Dirac field: $\psi(x)$: $\mathcal{L} = \bar{\psi}[i\gamma^\mu\partial_\mu - m]\psi$

Gauge fields: $A_\mu(x)$ ~ vector field,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i\not{D} - m]\psi$$

Real Scalar Field.

Earlier in the class you must have seen that if we treat K-G equation as the equation for a single-particle wave function $\varphi(x)$ (just like

Schroedinger eqn, except now relativistic)), then we have a horde of problems:

(i) as the energy $= \pm\sqrt{\vec{k}^2+m^2}$ $\Rightarrow$ can have free particles with negative energy!?

(ii) particle propagation $\langle \vec{x} | e^{-it\hat{H}} | \vec{y} \rangle$ from point $\vec{y}$ to pt. $\vec{x}$ is <u>acausal</u> ~ can find the particle outside the light-cone $\Rightarrow$ it would propagate "faster" than light...

~ in general we know that relativistic kinematics allows for a particle to decay into several particles $\Rightarrow$ particle # is not conserved $\Rightarrow$ should not have a single-particle wave function interpretation.

$\Rightarrow$ we quantize the system treating $\varphi(x)$ as a <u>field</u>!

Again, let us draw an analogy with Quantum Mechanics. Start with a system with degrees of freedom q_i described by Lagrangian $L(q_i, \dot{q}_i)$ $i=1,\dots,N$. Define <u>canonical momenta</u> $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$\Rightarrow$ get Hamiltonian $H = \sum_i \dot{q}_i p_i - L \Rightarrow H(q_i, p_i)$

H generates time evolution

$\Rightarrow$ quantize by "promoting" q_i, p_i to operators

$\hat{q}_i, \hat{p}_i$ such that $\left([\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \sim \text{commutator} \right)$

$$\boxed{[\hat{q}_i, \hat{p}_j] = i\,\delta_{ij}\,, \quad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0}$$

Quantum Mechanics		Field Theory
q_i	$\rightarrow$	$\varphi(x)$
i	$\rightarrow$	x^μ
$p_i = \frac{\partial L}{\partial \dot{q}_i}$	$\rightarrow$	$\pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)}$
$[\hat{q}_i, \hat{p}_j] = i\,\delta_{ij}$	$\rightarrow$	$[\varphi(\vec{x}, t), \pi(\vec{x}', t)] = i\,\delta(\vec{x} - \vec{x}')$
$[\hat{q}_i, \hat{q}_j] = 0$	$\rightarrow$	$[\varphi(\vec{x}, t), \varphi(\vec{x}', t)] = 0$
$[\hat{p}_i, \hat{p}_j] = 0$	$\rightarrow$	$[\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0.$

($\Rightarrow$ Note that canonical quantization favors time direction and is therefore not relat. inv. (physics is indeed invariant).)

$\hat{H}(\hat{q}_i, \hat{p}_i) \quad \rightarrow \quad \hat{H} = \int d^3x \cdot \mathcal{H}(\varphi, \pi)$

Hamiltonian gives time-evolution of the system

Canonical quantization: promote fields into operators with the equal-time commutation relations

$$\begin{cases} [\varphi(\vec{x},t),\ \pi(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y}) \\ [\varphi(\vec{x},t),\ \varphi(\vec{y},t)] = 0 \\ [\pi(\vec{x},t),\ \pi(\vec{y},t)] = 0\ . \end{cases}$$

In addition, require that time evolution is given by the Hamiltonian $\hat{H}$. For local operators this means

$$-i\,\frac{d\hat{O}}{dt} = [\hat{H},\hat{O}] \qquad \text{(Heisenberg representation)}$$

Interaction picture: $H = H_0 + H_{int}$ ← Schroedinger or Heisenberg pict. =>

$$\phi_I(\vec{x}, t) = e^{iH_0 t}\,\phi_S(\vec{x})\,e^{-iH_0 t},$$

$$i\frac{d}{dt}|\psi, t\rangle_I = \hat{H}_I |\psi, t\rangle_I \,; \quad \hat{H}_I = e^{i\hat{H}_0 t}\hat{H}_{int}\,e^{-i\hat{H}_0 t}.$$

$$= e^{iH_0 t}|\psi, t\rangle_S$$

Hamiltonian for real scalar theory with $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2$

is
$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2\right].$$

Generates time evolution: $\partial_0 \pi = i[H, \pi]$ (Heisenberg picture)

$\pi = \dot{\varphi} \Rightarrow \partial_0^2\varphi = i[H, \pi] = i\left[\int d^3y \left[\frac{\pi^2(y)}{2} + \frac{1}{2}(\vec{\nabla}\varphi(y))^2 + \right.\right.$

$\left.\left. + \frac{m^2}{2}\varphi^2(y)\right], \pi(x)\right]$, where $x^\mu = (t, \vec{x})$, $y^\mu = (t, \vec{y})$.

$[\pi^2, \pi] = 0$ as π's commute.

$[\varphi^2(y), \pi(x)]$: use $[A, BC] = [A, B]C + B[A, C]$

to write

$[\varphi^2(y), \pi(x)] = -[\pi(x), \varphi^2(y)] = -[\pi(x), \varphi(y)]\varphi(y) -$

$-\varphi(y)[\pi(x), \varphi(y)] = i\delta(\vec{x}-\vec{y})\varphi(y) + i\delta(\vec{x}-\vec{y})\varphi(y) =$

$= 2i\,\varphi(x)\,\delta(\vec{x}-\vec{y})$

The remaining commutator is a bit more subtle:

$$[(\vec{\nabla}\varphi(y))^2, \pi(x)] = -[\pi(x), (\vec{\nabla}\varphi(y))^2] =$$

$$= -[\pi(x), \vec{\nabla}\varphi(y)]\cdot\vec{\nabla}\varphi(y) - \vec{\nabla}\varphi(y)\cdot[\pi(x), \vec{\nabla}\varphi(y)] =$$

$$= \vec{\nabla}_y\left([\varphi(y), \pi(x)]\right)\cdot\vec{\nabla}\varphi(y) + \vec{\nabla}\varphi(y)\cdot\vec{\nabla}_y\left([\varphi(y), \pi(x)]\right)$$

$$= 2i\left[\vec{\nabla}_y\,\delta(\vec{x}-\vec{y})\right]\cdot\vec{\nabla}\varphi(y).$$

Hence we get

$$\partial_0^2\varphi = i\int d^3y\left[\frac{1}{\cancel{2}}\cdot\cancel{2}i\left(\vec{\nabla}_y\,\delta(\vec{x}-\vec{y})\right)\cdot\vec{\nabla}\varphi(y) + \frac{m^2}{\cancel{2}}\cdot\cancel{2}i\,\varphi(x)\delta(\vec{x}-\vec{y})\right]$$

$$= +\int d^3y\;\delta(\vec{x}-\vec{y})\,\vec{\nabla}^2\varphi(y) - \varphi(x)m^2 =$$

$$= \vec{\nabla}^2\varphi - m^2\varphi$$

$$\Rightarrow \left[\partial_0^2 - \vec{\nabla}^2 + m^2\right]\varphi = 0$$ $\Rightarrow$ Klein-Gordon eq'n holds at the operator level!

Hence we write for a solution of KG equation

$$\varphi(\vec{x},t) = \int\frac{d^3k}{(2\pi)^3\,2\varepsilon_k}\left[\hat{a}_{\vec{k}}\,e^{-ik\cdot x} + \hat{a}^+_{\vec{k}}\,e^{ik\cdot x}\right]$$

One can show that:

$$[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = (2\pi)^3\,2\varepsilon_k\,\delta^3(\vec{k}-\vec{k}')$$

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = [\hat{a}^+_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = 0$$

We have:

$$[\varphi(\vec{x},t), \pi(\vec{x}',t)] = i\,\delta^3(\vec{x}-\vec{x}')$$

$$[\varphi(\vec{x},t), \varphi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$$

(Equal-time commutation relations.)

where φ, π are operators, $\boxed{\pi(x) \equiv \frac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)}}$

Write

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k}\left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^+_{\vec{k}}\, e^{ik\cdot x}\right]$$

(now $\hat{a}$, $\hat{a}^+$ are <u>operators</u>)

$$\Rightarrow [\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = (2\pi)^3\, 2\mathcal{E}_k\, \delta^3(\vec{k}-\vec{k}')$$

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = [\hat{a}^+_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = 0$$

$\Leftarrow \begin{cases} \text{for } \mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \ldots \\ \pi = \partial_0\varphi = \dot{\varphi}. \end{cases}$

(can show) $\to$ see below

The Hamiltonian is $\boxed{H = \int d^3x\left[\dot{\varphi}(x)\,\pi(x) - \mathcal{L}\right]}$

$$\Rightarrow \quad i\frac{\partial}{\partial t}\hat{O} = [\hat{O}, \hat{H}]$$

Heisenberg picture.

$$\Rightarrow \hat{O}(\vec{x},t) = e^{i\hat{H}t}\,\hat{O}(\vec{x},0)\,e^{-i\hat{H}t}$$

Free scalar field: $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \Rightarrow$

$$\Rightarrow \pi(x) = \frac{\delta\mathcal{L}}{\delta\dot{\varphi}} = \frac{\delta\mathcal{L}}{\delta(\partial_0\varphi)} = \partial_0\varphi \quad \Rightarrow$$

$$H = \int d^3x\left[\dot{\varphi}^2 - \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2\right]$$

define $\overleftrightarrow{\partial_0}$ by $\varphi_1 \overleftrightarrow{\partial_0} \varphi_2 = \varphi_1 \partial_0 \varphi_2 - \varphi_2 \partial_0 \varphi_1$

Note that $\int d^3x \; e^{ik\cdot x} \overleftrightarrow{\partial_0} e^{-ik'\cdot x} = \int d^3x \left[-i\mathcal{E}_{k'} - i\mathcal{E}_k\right] \cdot e^{ix\cdot(k-k')} = -2i\,\mathcal{E}_k (2\pi)^3 \delta(\vec{k}-\vec{k}')$

$\Rightarrow$ if $\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \left[\hat{a}_{\vec{k}} e^{-ik\cdot x} + \hat{a}^+_{\vec{k}} e^{ik\cdot x}\right]$

$$\Rightarrow \int d^3x \; e^{ik'\cdot x} \overleftrightarrow{\partial_0} \varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \hat{a}_{\vec{k}} (-2i\mathcal{E}_k)(2\pi)^3 \delta(\vec{k}-\vec{k}')$$

$$= -i\,\hat{a}_{\vec{k}'}$$

as $\int d^3x \; e^{ik\cdot x} \overleftrightarrow{\partial_0} e^{+ik'\cdot x} = 0$. (why?)

$$\Rightarrow \boxed{\hat{a}_{\vec{k}} = \int d^3x \; e^{ik\cdot x} \, i\overleftrightarrow{\partial_0} \varphi(x)}$$

Similarly $\boxed{\hat{a}^+_{\vec{k}} = \int d^3x \; \varphi(x) \, i\overleftrightarrow{\partial_0} e^{-ik\cdot x}}$

$$\Rightarrow \left[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}\right] = \int d^3x \, d^3y \left[e^{ik\cdot x} i\overleftrightarrow{\partial_0}\varphi(x), \varphi(y) i\overleftrightarrow{\partial_0} e^{-ik'\cdot y}\right]$$

$$= \int d^3x\, d^3y \; e^{ik\cdot x - ik'\cdot y} (-) \left[\pi(x) - i\mathcal{E}_k \varphi(x), -i\mathcal{E}_{k'}\varphi(y) - \pi(y)\right]$$

$$= \int d^3x\, d^3y \; e^{ik\cdot x - ik'\cdot y} (-i)\left\{\left[\varphi(\overset{\vec{x},t}{\overset{\shortparallel}{x}}), \pi(\overset{\vec{y},t}{\overset{\shortparallel}{y}})\right]\mathcal{E}_k + \left[\varphi(y), \pi(x)\right]\mathcal{E}_{k'}\right\} =$$

$$= \int d^3x\, d^3y \; e^{ik\cdot x - ik'\cdot y} (\mathcal{E}_k + \mathcal{E}_{k'}) \delta(\vec{x}-\vec{y}) = e^{i(\mathcal{E}_k - \mathcal{E}_{k'})\cdot t} (\mathcal{E}_k + \mathcal{E}_{k'}) (2\pi)^3 \delta(\vec{k}-\vec{k}')$$

$= 2 \mathcal{E}_k (2\pi)^3 \delta(\vec{k}-\vec{k}')$ as advertised!

If $$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \left[\hat{a}_{\vec{k}} e^{-ik\cdot x} + \hat{a}^+_{\vec{k}} e^{ik\cdot x} \right]$$

then $$\pi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \left[-i\mathcal{E}_k \hat{a}_{\vec{k}} e^{-ik\cdot x} + i\mathcal{E}_k \hat{a}^+_{\vec{k}} e^{ik\cdot x} \right].$$

We'll be working in <u>Heisenberg</u> picture:

$$\left.\begin{aligned} \phi(\vec{x},t) &= e^{iHt} \phi(\vec{x},0) e^{-iHt} \\ \pi(\vec{x},t) &= e^{iHt} \pi(\vec{x},0) e^{-iHt} \end{aligned}\right\}$$ – operators are time-dependent

– $|\psi\rangle$ ~ states are time-indep.

There is also <u>Schrödinger</u> picture:

– operators are time-independent: $\phi_S(\vec{x}), \pi_S(\vec{x})$

– states are time-dependent $|\psi, t\rangle$.

<u>Interaction</u> picture mixes the two:

$$\phi_I(\vec{x},t) = e^{iH_0 t} \phi_S(\vec{x}) e^{-iH_0 t}$$

$$|a,t\rangle_I = e^{iH_0 t} |a,t\rangle_S = e^{iH_0 t} e^{-iHt} |a\rangle_H$$

H_0 ~ free (non-interacting) Hamiltonian.

in Sch. picture $[\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$.

in H. picture $i\dfrac{d\hat{O}}{dt} = [\hat{O}, \hat{H}]$

In general in quantum theory time-evolution is given by Hamiltonian:

$$\boxed{-i\hbar \frac{d}{dt} \langle \psi | \hat{O} | \phi \rangle = \langle \psi | [\hat{H}, \hat{O}] | \phi \rangle .}$$

(comes from generalizing Poisson bracket in QM)

We can separate this equation into two:

$$-i\hbar \frac{d\hat{O}}{dt} = [\hat{H}_1, O] \quad \text{and} \quad i\hbar \frac{d}{dt} |\psi\rangle = \hat{H}_2 |\psi\rangle$$

(+same for $|\phi\rangle$).

where $\hat{H} = \hat{H}_1 + \hat{H}_2$

<u>Heisenberg</u>: $\hat{H}_1 = \hat{H}$, $\hat{H}_2 = 0$

<u>Schrödinger</u>: $\hat{H}_1 = 0$, $\hat{H}_2 = \hat{H}$

<u>Interaction</u> picture: $\hat{H}_1 = \hat{H}_0$, $\hat{H}_2 = \hat{H}_I$

free Ham. interacting Ham. in interaction picture

(for more on these see Sterman, Appendix A.)

$\hat{H} = \hat{H}(\varphi, \pi)$ no explicit time dependence

For free scalar field we have

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 \right]$$

Plug in φ, $\pi = \dot{\varphi}$ $\Rightarrow$

$$H = \int d^3x \cdot \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \frac{d^3k'}{(2\pi)^3 2\mathcal{E}_{k'}} \Big\{ \hat{a}_{\vec{k}}\, \hat{a}_{\vec{k}'}\, e^{-ik\cdot x - ik'\cdot x} \cdot$$

$$\cdot \left[-\frac{\mathcal{E}_k \mathcal{E}_{k'}}{2} - \frac{1}{2}\vec{k}\cdot\vec{k}' + \frac{m^2}{2} \right] + \hat{a}^+_{\vec{k}}\, \hat{a}^+_{\vec{k}'}\, e^{ik\cdot x + ik'\cdot x} \cdot$$

$$\cdot \left[-\frac{\mathcal{E}_k \mathcal{E}_{k'}}{2} - \frac{\vec{k}\cdot\vec{k}'}{2} + \frac{m^2}{2} \right] + \hat{a}_{\vec{k}}\, \hat{a}^+_{\vec{k}'}\, e^{-ik\cdot x + ik'\cdot x} \cdot$$

$$\cdot \left[\frac{\mathcal{E}_k \mathcal{E}_{k'}}{2} + \frac{\vec{k}\cdot\vec{k}'}{2} + \frac{m^2}{2} \right] + \hat{a}^+_{\vec{k}}\, \hat{a}_{\vec{k}'}\, e^{ik\cdot x - ik'\cdot x} \cdot$$

$$\cdot \left[\frac{\mathcal{E}_k \mathcal{E}_{k'}}{2} + \frac{\vec{k}\cdot\vec{k}'}{2} + \frac{m^2}{2} \right] \Big\} =$$

integrate d^3x $\Rightarrow$ get $(2\pi)^3 \delta(\vec{k}+\vec{k}')$ for 1^{st} 2 terms and $(2\pi)^3 \delta(\vec{k}-\vec{k}')$ for last 2 terms.

$$= \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \frac{d^3k'}{(2\pi)^3 2\mathcal{E}_{k'}} \Big\{ \Big[\hat{a}_{\vec{k}}\, \hat{a}_{-\vec{k}}\, e^{-i\mathcal{E}_k t - i\mathcal{E}_k t} + \hat{a}^+_{\vec{k}}\, \hat{a}^+_{-\vec{k}} \cdot$$

$$\cdot e^{2i\mathcal{E}_k t} \Big] \underbrace{\left[-\frac{1}{2}\mathcal{E}_k^2 + \frac{1}{2}(\vec{k}^2 + m^2) \right]}_{=0} (2\pi)^3 \delta(\vec{k}+\vec{k}') +$$

$$+ \left[\hat{a}_{\vec{k}}\, \hat{a}^+_{\vec{k}} + \hat{a}^+_{\vec{k}}\, \hat{a}_{\vec{k}} \right] \Big[\frac{1}{2}\mathcal{E}_k^2 + \frac{1}{2}\overbrace{(\vec{k}^2 + m^2)}^{\mathcal{E}_k^2} \Big] (2\pi)^3 \delta(\vec{k}-\vec{k}') \Big\} =$$

$=$ (integrating over $\vec{k}'$ is trivial) $=$

$$= \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \frac{1}{2\cancel{\varepsilon_k}} \cdot \varepsilon_k^{\cancel{2}} \cdot \left[\hat{a}_{\vec{k}} \hat{a}^+_{\vec{k}} + \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} \right]$$

Finally, $$H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \frac{\varepsilon_k}{2} \left[\hat{a}_{\vec{k}} \hat{a}^+_{\vec{k}} + \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} \right]$$

aside:

Note that while φ, π were time-dependent (and hence in Heisenberg picture), $\hat{a}_{\vec{k}}$ & $\hat{a}^+_{\vec{k}}$ are not and are in Schroedinger picture.

One can show that $$\hat{a}^H_{\vec{k}}(t) = e^{iHt} \hat{a}^S_{\vec{k}} e^{-iHt} = e^{-i\varepsilon_k t} \hat{a}^S_{\vec{k}}$$

and $$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \left[\hat{a}^H_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}} + \hat{a}^{H+}_{\vec{k}}(t) e^{-i\vec{k}\cdot\vec{x}} \right].$$

Now, as $[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = (2\pi)^3\, 2\varepsilon_k\, \delta^3(\vec{k}-\vec{k}')$, we have

$$H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \varepsilon_k \left[\hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + \underbrace{\frac{1}{2} \cdot (2\pi)^3\, 2\varepsilon_k\, \delta^3(\vec{0})}_{\text{bad infinity!}} \right]$$

$$\Rightarrow H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \varepsilon_k \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + \infty.$$

$\infty \sim$ just a constant (for $\forall \vec{k}$) $\Rightarrow$ drop (zero point energy).

only gravity would see this $\infty \Rightarrow$ don't talk about it here

Def. Particle number operator $\hat{N}(\vec{k}) \equiv \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}}$

Write
$$\boxed{H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\, \varepsilon_k \hat{N}(\vec{k})}$$

Hence H = Net Energy = energy of one particle $\otimes$ # particles.

Def.
Total # of particles $\hat{N} \equiv \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \hat{N}(\vec{k})$.

Classify all states by eigenvalues of $\hat{N}$:

$$\hat{N}|n\rangle = n|n\rangle$$

Now, $[\hat{N}, \hat{a}^+_{\vec{k}}] = \int \frac{d^3k'}{(2\pi)^3\, 2\varepsilon_{k'}} [\hat{a}^+_{\vec{k}'} \hat{a}_{\vec{k}'}, \hat{a}^+_{\vec{k}}] =$

$$= \int \frac{d^3k'}{(2\pi)^3\, 2\varepsilon_{k'}} \left[\hat{a}^+_{\vec{k}'} \underbrace{\hat{a}_{\vec{k}'} \hat{a}^+_{\vec{k}}}_{= \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}'} + (2\pi)^3\, 2\varepsilon_k\, \delta^3(\vec{k}-\vec{k}')} - \hat{a}^+_{\vec{k}} \hat{a}^+_{\vec{k}'} \hat{a}_{\vec{k}'}\right] = \hat{a}^+_{\vec{k}}.$$

Hence
$$[\hat{N}, \hat{a}^+_{\vec{k}}] = \hat{a}^+_{\vec{k}}$$
$$[\hat{N}, \hat{a}_{\vec{k}}] = -\hat{a}_{\vec{k}}$$
← can be also shown.

$$\hat{N}\,\hat{a}^+_{\vec{k}}\,|n\rangle = \left(\hat{a}^+_{\vec{k}}\,\hat{N} + \hat{a}^+_{\vec{k}}\right)|n\rangle = (n+1)\,\hat{a}^+_{\vec{k}}\,|n\rangle$$

$\Rightarrow$ state $\hat{a}^+_{\vec{k}}|n\rangle$ has $(n+1)$-particles $\Rightarrow$

$\Rightarrow$ $\hat{a}^+_{\vec{k}}$ is a creation operator for a particle of momentum $\vec{k}$ & energy $\mathcal{E}_k$

$\hat{a}_{\vec{k}}$ is an annihilation operator — " —

as $\hat{N}\,\hat{a}_{\vec{k}}\,|n\rangle = \left(\hat{a}_{\vec{k}}\,\hat{N} - \hat{a}_{\vec{k}}\right)|n\rangle = (n-1)\,\hat{a}_{\vec{k}}\,|n\rangle$

\# particles ≥ 0 $\Rightarrow$ $\langle n|\,\underset{n\langle n|n\rangle}{\hat{N}_{\vec{k}}}\,|n\rangle \geq 0$ (if $n \geq 0$)

$\left(\langle n|\hat{N}|n\rangle = \int \frac{d^3k}{\cdots}\langle n|\hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}}|n\rangle \geq 0\right.$

(as $\langle n|\,\hat{a}^+_{\vec{k}}\,\hat{a}_{\vec{k}}\,|n\rangle \geq 0$) $\Rightarrow$ $n(\vec{k})\,\langle n|n\rangle \geq 0$

as $a_{\vec{k}}$ turns $n \to n-1$ $\Rightarrow$ there must be a ground state, otherwise would get $n<0$.

$$\hat{a}_{\vec{k}}\,|0\rangle = 0$$

← ground state (vacuum) (for any $\vec{k}$)

$\hat{N}_{\vec{k}}|0\rangle = 0$ $\Rightarrow$ $\hat{N}\,|0\rangle = 0$. ← zero particles in ground state

$$|\vec{k}\rangle = \hat{a}^+_{\vec{k}}\,|0\rangle$$

single-particle state

$$\langle\vec{k}'|\vec{k}\rangle = \langle 0|\,\hat{a}_{\vec{k}'}\,\hat{a}^+_{\vec{k}}\,|0\rangle = (2\pi)^3\,2\mathcal{E}_k\,\delta^3(\vec{k}-\vec{k}')$$

normalization