Last time | **Canonical Quantization (cont'd)**

Real Scalar Field (cont'd)

Canonical quantization:

- equal-time commutation relations:

$$\begin{cases} [\varphi(\vec{x},t), \pi(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y}) \\ [\varphi(\vec{x},t), \varphi(\vec{y},t)] = 0 = [\pi(\vec{x},t), \pi(\vec{y},t)] \end{cases}$$

- time evolution is generated by the Hamiltonian:

$$\boxed{-i\frac{d\hat{O}}{dt} = [\hat{H}, \hat{O}]}$$

$$\Downarrow$$

$$\boxed{[\Box + m^2]\hat{\varphi}(x) = 0}$$ EOM satisfied by the field operator!

$$\boxed{\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2E_k}\left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^{+}_{\vec{k}}\, e^{ik\cdot x}\right]}$$

with

$$[\hat{a}_{\vec{k}}, \hat{a}^{+}_{\vec{k}'}] = (2\pi)^3 2E_k\, \delta^3(\vec{k}-\vec{k}')$$

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0 = [\hat{a}^{+}_{\vec{k}}, \hat{a}^{+}_{\vec{k}'}].$$

The Hamiltonian

$$H = \int d^3x \left[\frac{\pi^2}{2} + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 \right]$$

$$\Rightarrow \boxed{H = \int \frac{d^3k}{(2\pi)^3 2E_k} E_k \, \hat{a}^+_{\vec{k}} \, \hat{a}_{\vec{k}}}$$

~ sum of energies of all particles

~ clear physical meaning

$= 2\,\mathcal{E}_k\,(2\pi)^3\,\delta(\vec{k}-\vec{k}')$ as advertised!

If $\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k}\left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^{+}_{\vec{k}}\, e^{ik\cdot x}\right]$

then $\pi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k}\left[-i\,\mathcal{E}_k\,\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + i\,\mathcal{E}_k\,\hat{a}^{+}_{\vec{k}}\, e^{ik\cdot x}\right]$.

We'll be working in <u>Heisenberg</u> picture:

$$\left.\begin{aligned}\phi(\vec{x},t) &= e^{iHt}\,\phi(\vec{x},0)\,e^{-iHt}\\ \pi(\vec{x},t) &= e^{iHt}\,\pi(\vec{x},0)\,e^{-iHt}\end{aligned}\right\}$$

– operators are time-dependent

– $|\psi\rangle$ ~ states are time-indep.

There is also <u>Schrödinger</u> picture:

– operators are time-independent: $\phi_S(\vec{x}),\ \pi_S(\vec{x})$

– states are time-dependent $|\psi, t\rangle$.

<u>Interaction</u> picture mixes the two:

$$\phi_I(\vec{x},t) = e^{iH_0 t}\,\phi_S(\vec{x})\,e^{-iH_0 t}$$

$$|a,t\rangle_I = e^{iH_0 t}\,|a,t\rangle_S = e^{iH_0 t}\,e^{-iHt}\,|a\rangle_H$$

H_0 ~ free (non-interacting) Hamiltonian.

in Sch. picture $[\phi(\vec{x}), \pi(\vec{y})] = i\,\delta^3(\vec{x}-\vec{y})$.

in H. picture $i\frac{d\hat{O}}{dt} = [\hat{O}, \hat{H}]$

In general in quantum theory time-evolution is given by Hamiltonian:

$$\boxed{-i\hbar \frac{d}{dt} \langle \psi | \hat{O} | \phi \rangle = \langle \psi | [\hat{H}, \hat{O}] | \phi \rangle}$$

(comes from generalizing Poisson bracket in QM)

We can separate this equation into two:

$$-i\hbar \frac{d\hat{O}}{dt} = [\hat{H}_1, O] \quad \text{and} \quad i\hbar \frac{d}{dt} |\psi\rangle = \hat{H}_2 |\psi\rangle$$

(+same for $|\phi\rangle$).

where $\hat{H} = \hat{H}_1 + \hat{H}_2$

<u>Heisenberg</u>: $\hat{H}_1 = \hat{H}$, $\hat{H}_2 = 0$

<u>Schrödinger</u>: $\hat{H}_1 = 0$, $\hat{H}_2 = \hat{H}$

<u>Interaction</u> picture: $\hat{H}_1 = \hat{H}_0$, $\hat{H}_2 = \hat{H}_I$

free Ham. — interacting Ham. in interaction picture

(for more on these see Sterman, Appendix A.)

$\hat{H} = \hat{H}(\varphi, \pi)$ no explicit time dependence

For free scalar field we have

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 \right]$$

Plug in φ, $\pi = \dot{\varphi}$ $\Rightarrow$

$$H = \int d^3x \cdot \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}} \Big\{ \hat{a}_{\vec{k}} \hat{a}_{\vec{k}'} e^{-ik\cdot x - ik'\cdot x} \cdot$$

$$\cdot \left[-\frac{\varepsilon_k \varepsilon_{k'}}{2} - \frac{1}{2}\vec{k}\cdot\vec{k}' + \frac{m^2}{2} \right] + \hat{a}^+_{\vec{k}} \hat{a}^+_{\vec{k}'} e^{ik\cdot x + ik'\cdot x} \cdot$$

$$\cdot \left[-\frac{\varepsilon_k \varepsilon_{k'}}{2} - \frac{\vec{k}\cdot\vec{k}'}{2} + \frac{m^2}{2} \right] + \hat{a}_{\vec{k}} \hat{a}^+_{\vec{k}'} e^{-ik\cdot x + ik'\cdot x} \cdot$$

$$\left[\frac{\varepsilon_k \varepsilon_{k'}}{2} + \frac{\vec{k}\cdot\vec{k}'}{2} + \frac{m^2}{2} \right] + \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}'} e^{ik\cdot x - ik'\cdot x} \cdot$$

$$\cdot \left[\frac{\varepsilon_k \varepsilon_{k'}}{2} + \frac{\vec{k}\cdot\vec{k}'}{2} + \frac{m^2}{2} \right] \Big\} =$$

integrate $d^3x \Rightarrow$ get $(2\pi)^3 \delta(\vec{k}+\vec{k}')$ for 1st 2 terms and $(2\pi)^3 \delta(\vec{k}-\vec{k}')$ for last 2 terms.

$$= \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}} \Big\{ \Big[\hat{a}_{\vec{k}} \hat{a}_{-\vec{k}} e^{-i\varepsilon_k t - i\varepsilon_k t} + \hat{a}^+_{\vec{k}} \hat{a}^+_{-\vec{k}} \cdot$$

$$\cdot e^{2i\varepsilon_k t} \Big] \underbrace{\left[-\frac{1}{2}\varepsilon_k^2 + \frac{1}{2}\left(\vec{k}^2 + m^2\right) \right]}_{=0} (2\pi)^3 \delta(\vec{k}+\vec{k}') +$$

$$+ \left[\hat{a}_{\vec{k}} \hat{a}^+_{\vec{k}} + \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} \right] \left[\frac{1}{2}\varepsilon_k^2 + \frac{1}{2}\overbrace{\left(\vec{k}^2 + m^2\right)}^{\varepsilon_k^2} \right] (2\pi)^3 \delta(\vec{k}-\vec{k}') \Big\} =$$

= (integrating over $\vec{k}'$ - trivial) =

$$= \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \frac{1}{2\cancel{\varepsilon_k}} \cdot \varepsilon_k^{\cancel{2}} \cdot \left[\hat{a}_{\vec{k}} \hat{a}^+_{\vec{k}} + \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} \right]$$

Finally, $$H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \frac{\varepsilon_k}{2} \left[\hat{a}_{\vec{k}} \hat{a}^+_{\vec{k}} + \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} \right]$$

aside:

Note that while φ, π were time-dependent (and hence in Heisenberg picture), $\hat{a}_{\vec{k}}$ & $\hat{a}^+_{\vec{k}}$ are not and are in Schroedinger picture.

One can show that $$\hat{a}^H_{\vec{k}}(t) = e^{iHt}\, \hat{a}^S_{\vec{k}}\, e^{-iHt} =$$
$$= e^{-i\varepsilon_k t}\, \hat{a}^S_{\vec{k}}$$

and $$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \left[\hat{a}^H_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}} + \hat{a}^{+H}_{\vec{k}}(t)\, e^{-i\vec{k}\cdot\vec{x}} \right].$$

Now, as $[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = (2\pi)^3\, 2\varepsilon_k\, \delta^3(\vec{k}-\vec{k}')$, we have

$$H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \varepsilon_k \left[\hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + \underbrace{\frac{1}{2} \cdot (2\pi)^3\, 2\varepsilon_k\, \delta^3(\vec{0})}_{\text{bad infinity!}} \right]$$

$$\Rightarrow H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \varepsilon_k\, \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + \infty.$$

$\infty \sim$ just a constant (for $\forall \vec{k}$) $\Rightarrow$ drop (zero point energy).

only gravity would see this ∞ $\Rightarrow$ don't talk about it here

Def. Particle number operator $\left(\hat{N}(\vec{k}) \equiv \hat{a}^{+}_{\vec{k}}\, \hat{a}_{\vec{k}}\right)$

Write $$\boxed{H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\ \varepsilon_k\, \hat{N}(\vec{k})}$$

Hence $H =$ Net Energy $=$ energy of one particle $\otimes$ # particles.

Def.
Total # of particles $\hat{N} \equiv \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\ \hat{N}(\vec{k})$.

Classify all states by eigenvalues of $\hat{N}$:

$$\hat{N}\,|n\rangle = n\,|n\rangle$$

Now, $[\hat{N}, \hat{a}^{+}_{\vec{k}}] = \int \frac{d^3k'}{(2\pi)^3\, 2\varepsilon_{k'}}\ [\hat{a}^{+}_{\vec{k}'}\hat{a}_{\vec{k}'}, \hat{a}^{+}_{\vec{k}}] =$

$$= \int \frac{d^3k'}{(2\pi)^3\, 2\varepsilon_{k'}} \left[\hat{a}^{+}_{\vec{k}'}\, \underbrace{\hat{a}_{\vec{k}'}\, \hat{a}^{+}_{\vec{k}}}_{= \hat{a}^{+}_{\vec{k}}\, \hat{a}_{\vec{k}'} + (2\pi)^3\, 2\varepsilon_k\, \delta^3(\vec{k}-\vec{k}')} - \hat{a}^{+}_{\vec{k}}\, \hat{a}^{+}_{\vec{k}'}\, \hat{a}_{\vec{k}'}\right] = \hat{a}^{+}_{\vec{k}}.$$

Hence
$$\boxed{\begin{aligned} [\hat{N}, \hat{a}^{+}_{\vec{k}}] &= \hat{a}^{+}_{\vec{k}} \\ [\hat{N}, \hat{a}_{\vec{k}}] &= -\hat{a}_{\vec{k}} \end{aligned}}$$
← can be also shown.

$$\hat{N}\,\hat{a}^+_{\vec{k}}|n\rangle = \left(\hat{a}^+_{\vec{k}}\hat{N} + \hat{a}^+_{\vec{k}}\right)|n\rangle = (n+1)\,\hat{a}^+_{\vec{k}}|n\rangle$$

$\Rightarrow$ state $\hat{a}^+_{\vec{k}}|n\rangle$ has $(n+1)$-particles $\Rightarrow$

$\Rightarrow$ $\hat{a}^+_{\vec{k}}$ is a creation operator for a particle of momentum $\vec{k}$ & energy ε_k

$\hat{a}_{\vec{k}}$ is an annihilation operator –"–

as $\hat{N}\,\hat{a}_{\vec{k}}|n\rangle = \left(\hat{a}_{\vec{k}}\hat{N} - \hat{a}_{\vec{k}}\right)|n\rangle = (n-1)\,\hat{a}_{\vec{k}}|n\rangle$

particles $\geq 0 \Rightarrow \langle n|\hat{N}|n\rangle \geq 0$ $\overset{?}{\Rightarrow}$ $\overset{?}{\Rightarrow}$ $n \geq 0$)

(as $\langle n|\hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}}|n\rangle \geq 0 \Rightarrow$ $\langle n|\hat{N}|n\rangle = \int d^3k \ldots \langle n|\hat{a}^+_{\vec{k}}\hat{a}_{\vec{k}}|n\rangle \geq 0$, $\langle n|\hat{N}|n\rangle = n\langle n|n\rangle$ $\Rightarrow n\langle n|n\rangle \geq 0 \Rightarrow n \geq 0$

as $a_{\vec{k}}$ turns $n \to n-1 \Rightarrow$ there must be a ground state, otherwise would get $n<0$.

$$\hat{a}_{\vec{k}}|0\rangle = 0$$

– ground state (vacuum) (for any $\vec{k}$)

$\hat{N}_{\vec{k}}|0\rangle = 0 \Rightarrow \hat{N}|0\rangle = 0$. – zero particles in ground state

$$|\vec{k}\rangle = \hat{a}^+_{\vec{k}}|0\rangle$$

single-particle state

$$\langle\vec{k}'|\vec{k}\rangle = \langle 0|\hat{a}_{\vec{k}'}\hat{a}^+_{\vec{k}}|0\rangle = (2\pi)^3\, 2\varepsilon_k\, \delta^3(\vec{k}-\vec{k}')$$

normalization

$|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^+_{\vec{k}_1} \hat{a}^+_{\vec{k}_2} |0\rangle$ two-particle state

$H |\vec{k}_1, \vec{k}_2\rangle = (\mathcal{E}_{k_1} + \mathcal{E}_{k_2}) |\vec{k}_1, \vec{k}_2\rangle$

In general $|\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n\rangle = \hat{a}^+_{\vec{k}_1} \hat{a}^+_{\vec{k}_2} \ldots \hat{a}^+_{\vec{k}_n} |0\rangle$

n-particle state. (Fock states)

Any state of the theory can be expanded into Fock states:

$$\boxed{|\Psi\rangle = c_0 |0\rangle + \int \frac{d^3k}{(2\pi)^3 2E_k} c_{\vec{k}} |\vec{k}\rangle + \frac{1}{2!} \int \frac{d^3k_1\, d^3k_2}{(2\pi)^3 2E_{k_1} (2\pi)^3 2E_{k_2}} \cdot c_{\vec{k}_1, \vec{k}_2} |\vec{k}_1, \vec{k}_2\rangle + \ldots}$$

Wave functions:

$\psi(\vec{k}) = \langle \vec{k} | \Psi\rangle$ ~ single-particle wave function

$\Rightarrow \psi(\vec{k}) = \int \frac{d^3k'}{(2\pi)^3 2E_{k'}} c_{\vec{k}'} \langle \vec{k} | \vec{k}'\rangle = c_{\vec{k}}$

$\Rightarrow c_{\vec{k}}$ is the single-particle wave function (in momentum space)

$\psi(\vec{k}_1, \vec{k}_2) = \langle \vec{k}_1, \vec{k}_2 | \Psi\rangle$ ~ two-particle wave function

$\Rightarrow \psi(\vec{k}_1, \vec{k}_2) = c_{\vec{k}_1, \vec{k}_2}$ ~ can show

$c_{\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n}$ ~ n-particle wave function

Complex Scalar Field

$$\boxed{\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi}$$ ~ the Lagrangian (charged scalar field)

Obeys Klein-Gordon equation classically

$$\boxed{[\Box + m^2]\varphi = 0} \qquad \left(\frac{\delta \mathcal{L}}{\delta \varphi^*} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi^*)} = 0 \right)$$

General solution:

$$\boxed{\varphi(x) = \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k} \left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{b}^+_{\vec{k}}\, e^{ik\cdot x} \right]}$$

as φ is now complex $\Rightarrow$ $\hat{a}_{\vec{k}} \neq \hat{b}_{\vec{k}}$.

$$\boxed{\varphi^+(x) = \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k} \left[\hat{a}^+_{\vec{k}}\, e^{ik\cdot x} + \hat{b}_{\vec{k}}\, e^{-ik\cdot x} \right]}$$

Canonical momenta:

$$\pi_\varphi = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}} = \dot{\varphi}^* \quad ; \quad \pi_{\varphi^*} = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}^*} = \dot{\varphi}$$

Demand that according to the rules of canonical quantization:

$$\boxed{\begin{aligned} [\varphi(\vec{x},t), \pi_\varphi(\vec{y},t)] &= i\,\delta^3(\vec{x}-\vec{y}) \\ [\varphi^+(\vec{x},t), \pi_{\varphi^*}(\vec{y},t)] &= i\,\delta^3(\vec{x}-\vec{y}) \end{aligned}}$$

all other commutators are zero (equal time)

Quantization of Spinor Field

$$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m \right]\psi$$

First of all, let us change the representation: we will go from Weyl to (the standard) Dirac representation:

$$\psi_{\text{Weyl}} \to S\,\psi_{\text{Weyl}}, \text{ where } S = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \text{ in } 2\times 2 \text{ notation}$$

$S^T S = \mathbb{1}$ orthogonal matrix

$$\psi_{\substack{\text{Dirac}\\ \text{rep}}} = S\,\psi_{\substack{\text{weyl}\\ \text{rep}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R + \chi_L \\ \chi_R - \chi_L \end{pmatrix}.$$

$$\gamma^\mu_{\text{Weyl}} \to \gamma^\mu_{\text{Dirac}} = S\,\gamma^\mu_{\text{Weyl}}\,S^{-1} = S\,\gamma^\mu_{\text{weyl}}\,S^T$$

$$\gamma^0_{\text{Weyl}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i_{\text{weyl}} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \Rightarrow$$

$$\gamma^0_{\text{Dirac}} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \underbrace{\psi_w^+ S^T}_{\psi_D^+}\underbrace{S\gamma_w^0 S^T}_{\gamma_D^0}$$

$$\Rightarrow \gamma^0_{\text{Dirac}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \left[\mathcal{L} = \underbrace{\bar{\psi}_W S^T}_{\bar{\psi}_D} S\left[i\gamma^\mu \partial_\mu - m\right] S^T \underbrace{S\psi_W}_{\psi_D} = \bar{\psi}_{Dir}\left[i\gamma_D^\mu \partial_\mu - m\right]\psi_D. \right]$$

$$\gamma^i_{\text{Dirac}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \sigma^i & \sigma^i \\ -\sigma^i & \sigma^i \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2\sigma^i \\ -2\sigma^i & 0 \end{pmatrix} = \gamma^i_{Weyl}$$

$$\Rightarrow \boxed{\gamma^0_{Dirac} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \quad \gamma^i_{Dirac} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}.$$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in either representation (chirality operator)

$\gamma^5_{Dirac} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. ~ different from Weyl basis.

$\mathbb{P}_L = \frac{\mathbb{1} - \gamma^5}{2}$, $\mathbb{P}_R = \frac{\mathbb{1} + \gamma^5}{2}$

$$\Rightarrow P_L = \frac{1}{2}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \Rightarrow P_L \psi_{Dirac} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \chi_R + \chi_L \\ \chi_R - \chi_L \end{pmatrix} =$$

$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 2\chi_L \\ -2\chi_L \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_L \\ -\chi_L \end{pmatrix}$ ~ "removes" χ_R from ψ_{Dirac}, leaving only χ_L.

$$P_R = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow P_R \psi_{Dirac} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \chi_R + \chi_L \\ \chi_R - \chi_L \end{pmatrix}$$

$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 2\chi_R \\ 2\chi_R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix}$ ~ leaves only χ_R.

(all γ-matrix formulas work in both bases.)

Peskin ~ Weyl representation, Ryder ~ Dirac representation

$\gamma^5 \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_L \\ -\chi_L \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} \chi_L \\ -\chi_L \end{pmatrix} \Rightarrow$ chirality -1 ; $\gamma^5 \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix} \Rightarrow$ chirality $+1$.