Last time | Finished canonical quantization of the scalar field : $\hat{N}(\vec{k}) = \hat{a}^{+}_{\vec{k}} \hat{a}_{\vec{k}}$ ~ particle # operator

Defined the vacuum state $|0\rangle$ by $\hat{a}_{\vec{k}}|0\rangle = 0$.

1-particle state $|\vec{k}\rangle = \hat{a}^{+}_{\vec{k}}|0\rangle$

2-particle state $|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^{+}_{\vec{k}_1} \hat{a}^{+}_{\vec{k}_2}|0\rangle$

$\vdots \qquad\qquad \vdots$

General state in field theory is

$$|\psi\rangle = c_0|0\rangle + \int \frac{d^3k}{(2\pi)^3 2E_k} c_{\vec{k}} |\vec{k}\rangle + \frac{1}{2!}\int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \frac{d^3k_2}{(2\pi)^3 2E_{k_2}} \cdot c_{\vec{k}_1,\vec{k}_2} |\vec{k}_1, \vec{k}_2\rangle + \dots$$

$c_{\vec{k}}$ ~ single-particle wave function

$c_{\vec{k}_1,\vec{k}_2}$ ~ two-particle — " —

$\vdots$

Quantization of Spinor Field

$$\mathcal{L} = \bar{\psi}\,[\,i\gamma^\mu \partial_\mu - m\,]\,\psi$$

First of all, let us change the representation:
we will go from Weyl to (the standard) Dirac representation:

$$\psi_{\text{Weyl}} \to S\,\psi_{\text{Weyl}}, \text{ where } S = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \text{ in } 2\times 2 \text{ notation}$$

$S^T S = \mathbb{1}$ orthogonal matrix

$$\psi_{\substack{\text{Dirac}\\\text{rep}}} = S\,\psi_{\substack{\text{weyl}\\\text{rep}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R + \chi_L \\ \chi_R - \chi_L \end{pmatrix}.$$

$$\gamma^\mu_{\text{Weyl}} \to \gamma^\mu_{\text{Dirac}} = S\,\gamma^\mu_{\text{Weyl}}\,S^{-1} = S\,\gamma^\mu_{\text{weyl}}\,S^T$$

$$\gamma^0_{\text{Weyl}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i_{\text{weyl}} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \Rightarrow$$

$$\gamma^0_{\text{Dirac}} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \overbrace{\psi_w^+ S^T}^{\psi_D^+}\overbrace{S\,\gamma_w^0\,S^T}^{\gamma_D^0}$$

$$\Rightarrow \gamma^0_{\text{Dirac}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \left[\mathcal{L} = \underbrace{\bar{\psi}_w S^T}_{\bar{\psi}_D} S[i\gamma^\mu\partial_\mu - m]S^T \underbrace{S\psi_w}_{\psi_D} = \bar{\psi}_{Dir}\,[i\gamma_D^\mu\partial_\mu - m]\,\psi_D.\right]$$

$$\gamma^i_{\text{Dirac}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \sigma^i & \sigma^i \\ -\sigma^i & \sigma^i \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2\sigma^i \\ -2\sigma^i & 0 \end{pmatrix} = \gamma^i_{Weyl}$$

$$\Rightarrow \boxed{\gamma^0_{Dirac} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; ; \; \gamma^i_{Dirac} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}.$$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in either representation (chirality operator)

$\gamma^5_{Dirac} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. ~ different from Weyl basis.

$\mathbb{P}_L = \frac{\mathbb{1} - \gamma^5}{2}$, $\mathbb{P}_R = \frac{\mathbb{1} + \gamma^5}{2}$

$$\Rightarrow P_L = \frac{1}{2}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \Rightarrow P_L \psi_{Dirac} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \chi_R + \chi_L \\ \chi_R - \chi_L \end{pmatrix} =$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 2\chi_L \\ -2\chi_L \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_L \\ -\chi_L \end{pmatrix}$$ ~ "removes" χ_R from ψ_{Dirac}, leaving only χ_L.

$$P_R = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow P_R \psi_{Dirac} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \chi_R + \chi_L \\ \chi_R - \chi_L \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 2\chi_R \\ 2\chi_R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix}$$ ~ leaves only χ_R.

(all γ-matrix formulas work in both bases.)

Peskin ~ Weyl representation, Ryder ~ Dirac representation

$\gamma^5 \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_L \\ -\chi_L \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} \chi_L \\ -\chi_L \end{pmatrix} \Rightarrow$ chirality -1 ; $\gamma^5 \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix} \Rightarrow$ chirality $+1$.

Solution of Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \Rightarrow \text{apply } i\gamma^\nu \partial_\nu \quad \Rightarrow$$

$$[\underbrace{-\gamma^\nu \partial_\nu \gamma^\mu \partial_\mu}_{} - m\, i\gamma^\nu \partial_\nu]\psi = 0$$

$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu = \frac{1}{2}\{\gamma^\nu, \gamma^\mu\}\partial_\mu \partial_\nu = g^{\mu\nu}\partial_\mu \partial_\nu = \partial_\mu \partial^\mu$$

$$\Rightarrow [-\partial_\mu \partial^\mu - m \underbrace{i\gamma^\mu \partial_\mu}_{=m \text{ by Dirac equation}}]\psi = 0$$

$$\Rightarrow \boxed{[\partial_\mu \partial^\mu + m^2]\psi = 0}$$ (This is how we constructed the Dirac $\mathcal{L}$.)

$\Rightarrow$ if the field satisfies Dirac equation, it also satisfies Klein-Gordon equation! ($\Rightarrow$ also has < 0 energy "particles")

$\Rightarrow$ Write the solution as

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k}\left[e^{-ik\cdot x}\psi^{(+)}(\vec{k}) + e^{ik\cdot x}\psi^{(-)}(\vec{k})\right]$$

& plug back into the original Dirac equation:

$\partial_\mu \to -ik_\mu$ in the 1st term, $+ik_\mu$ in the second

$$\Rightarrow \text{get} \quad (\gamma\cdot k - m)\psi^{(+)}(\vec{k}) = 0$$

$$(\gamma\cdot k + m)\psi^{(-)}(\vec{k}) = 0$$

$$\Rightarrow \text{write} \quad \psi^{(+)} = \begin{pmatrix} \psi_u^{(+)} \\ \psi_\ell^{(+)} \end{pmatrix}$$

$$\Rightarrow \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \Rightarrow$$

$$\gamma \cdot k = \gamma^0 k_0 + \gamma^i k_i = \gamma_0 k_0 - \vec{\gamma} \cdot \vec{k} = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$$

$$- \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{k} \\ -\vec{\sigma} \cdot \vec{k} & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon & -\vec{\sigma} \cdot \vec{k} \\ \vec{\sigma} \cdot \vec{k} & -\varepsilon \end{pmatrix}$$

$$\Rightarrow (\gamma \cdot k - m)\psi^{(+)} = \begin{pmatrix} \varepsilon - m & -\vec{\sigma} \cdot \vec{k} \\ \vec{\sigma} \cdot \vec{k} & -\varepsilon - m \end{pmatrix} \begin{pmatrix} \psi_u^{(+)} \\ \psi_\ell^{(+)} \end{pmatrix} = 0$$

$$\begin{cases} (\varepsilon - m)\psi_u^{(+)} - \vec{\sigma} \cdot \vec{k}\, \psi_\ell^{(+)} = 0 \\ \vec{\sigma} \cdot \vec{k}\, \psi_u^{(+)} - (\varepsilon + m)\psi_\ell^{(+)} = 0 \end{cases} \Rightarrow \psi_\ell^{(+)} = \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \psi_u^{(+)}$$

~ solves the whole thing (why?)

$$\Rightarrow \psi^{(+)} = \begin{pmatrix} \psi_u^{(+)} \\ \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \psi_u^{(+)} \end{pmatrix}$$

$\Rightarrow$ reduced a 4-component unknown spinor to 2 unknown components

Similarly
$$\psi^{(-)} = \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \psi_\ell^{(-)} \\ \psi_\ell^{(-)} \end{pmatrix}.$$

Choose a basis: $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and

Define

$$\boxed{u_r(\vec{k}) = \sqrt{\varepsilon + m} \begin{pmatrix} \chi_r \\ \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \chi_r \end{pmatrix}; \quad v_r(\vec{k}) = \sqrt{\varepsilon + m} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \chi_r \\ \chi_r \end{pmatrix}} \quad r = 1, 2$$

then we write

$$\boxed{\psi(x) = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3\, 2\,\mathcal{E}_k} \left\{ \hat{b}_{\vec{k},r}\, u_r(\vec{k})\, e^{-ik\cdot x} + \hat{d}^{+}_{\vec{k},r}\, v_r(\vec{k})\, e^{ik\cdot x} \right\}}$$

Canonical quantization: $\pi = \frac{\delta \mathcal{L}}{\delta(\partial_0 \psi)} = \bar{\psi}\, i\gamma^0 =$

$= \psi^+ \gamma_0 \gamma^0 \cdot i = i\psi^+$ as $\gamma_0\gamma^0 = \mathbb{1}$.

promote $\hat{b}$ & $\hat{d}$ to operators (note that $(i\gamma^\mu\partial_\mu - m)\psi = 0$ still holds!)

$\Rightarrow H = \int d^3x \left[\pi\dot{\psi} - \mathcal{L}\right] = \int d^3x \cdot \left[i\psi^+\dot{\psi} - \mathcal{L}\right]$

$= \int d^3x \left[i\bar{\psi}\gamma^0\partial_0\psi - \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \right] =$

$= \int d^3x \left[\cancel{i\bar{\psi}\gamma^0 \partial_0\psi} - \cancel{\bar{\psi}\, i\gamma^0 \partial_0\psi} - i\bar{\psi}\gamma^i\partial_i\psi + \right.$

$\left. + \bar{\psi}\psi m \right] = \int d^3x\, \bar{\psi} \underbrace{\left[-i\gamma^i\partial_i + m\right]\psi}_{i\gamma^0\partial_0\psi \text{ (Dirac eqn.)}}$

$= \int d^3x\; i\psi^+\partial_0\psi \;\Rightarrow \boxed{H = \int d^3x\; i\psi^+\partial_0\psi}$

H is not ≥ 0 at the classical level ~ problem!

$\hookrightarrow$ this is cured by quantization!

Plug in the solution of Dirac equation into the Hamiltonian:

$$\psi^+ = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \left\{ \hat{b}^+_{\vec{k},r}\, u^+_r(\vec{k})\, e^{ik\cdot x} + \hat{d}_{\vec{k},r}\, v^+_r(\vec{k})\, e^{-ik\cdot x} \right\}$$

$$\Rightarrow H = \int d^3x\, i\psi^+ \partial_0 \psi = \int d^3x \sum_{r,r'=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}}\, i\cdot$$

$$\cdot \left[\hat{b}^+_{\vec{k}',r'}\, u^+_{r'}(\vec{k}')\, e^{ik'\cdot x} + \hat{d}_{\vec{k}',r'}\, v^+_{r'}(\vec{k}')\, e^{-ik'\cdot x} \right] \cdot \left[\hat{b}_{\vec{k},r}\, u_r(\vec{k}) \cdot \right.$$

$$\left. \cdot (-i\varepsilon_k)\, e^{-ik\cdot x} + \hat{d}^+_{\vec{k},r}\, v_r(\vec{k})\, (i\varepsilon_k)\, e^{ik\cdot x} \right]$$

① $\hat{b}^+\hat{b}$ -term: $\int d^3x\, e^{ik'\cdot x - ik\cdot x} = (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$

$$\Rightarrow \text{get} \sum_{r,r'=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{d^3k'}{(2\pi)^3 2\varepsilon_k}\, \varepsilon_k\, (2\pi)^3 \delta(\vec{k}-\vec{k}')\, \hat{b}^+_{\vec{k}',r'}\, \hat{b}_{\vec{k},r}$$

$$u^+_{r'}(\vec{k}')\, u_r(\vec{k}) = \sum_{r,r'=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{1}{2}\, \hat{b}^+_{\vec{k},r'}\, \hat{b}_{\vec{k},r}\, u^+_{r'}(\vec{k})\, u_r(\vec{k})$$

$$\Rightarrow u_r(\vec{k}) = \sqrt{\varepsilon_k + m} \begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon_k + m}\chi_r \end{pmatrix} \Rightarrow u^+_{r'} u_r = (\varepsilon_k + m)\left[\chi^T_{r'} \cdot \chi_r + \right.$$

$$\left. + \chi^T_{r'} \frac{\overset{=\vec{\sigma}}{\vec{\sigma}^+}\cdot\vec{k}}{\varepsilon_k + m} \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon_k + m}\chi_r \right] = (\varepsilon_k + m)\left[\delta_{rr'} + \frac{\vec{k}^2}{(\varepsilon_k + m)^2}\delta_{rr'} \right]$$

$$= \delta_{rr'} \frac{1}{\varepsilon_k + m}\left[(\varepsilon_k + m)^2 + \overbrace{\varepsilon_k^2 - m^2}^{\vec{k}^2} \right] = \delta_{rr'} \frac{1}{\varepsilon_k + m}\left(2\varepsilon_k^2 + 2\varepsilon_k m \right) =$$

$$= 2\varepsilon_k \delta_{rr'}$$

$$\Rightarrow \hat{b}^{+}\hat{b}\text{-term} = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \, \varepsilon_k \, \hat{b}^{+}_{\vec{k},r} \, \hat{b}_{\vec{k},r}.$$

(2) $\hat{b}^{+}\hat{d}^{+}$ term: $\int d^3x \, e^{ik'\cdot x + ik\cdot x} = e^{2i\varepsilon_k \cdot t} (2\pi)^3 \delta(\vec{k}+\vec{k}')$

$\Rightarrow$ get $\propto u^{+}_{r'}(-\vec{k}) \, v_r(\vec{k})$

$$v_r(\vec{k}) = \sqrt{\varepsilon_k + m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon_k + m} \chi_r \\ \chi_r \end{pmatrix} \Rightarrow u^{+}_{r'}(-\vec{k}) \, v_r(\vec{k}) =$$

$$= (\varepsilon_k + m) \begin{pmatrix} \chi^T_{r'} & \chi^T_{r'} \frac{\vec{\sigma}^{+}\cdot(-\vec{k})}{\varepsilon_k + m} \end{pmatrix} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon_k + m} \chi_r \\ \chi_r \end{pmatrix} \propto$$

$$= \chi^T_{r'} (\vec{\sigma}\cdot\vec{k}) \chi_r - \chi^T_{r'} (\vec{\sigma}\cdot\vec{k}) \chi_r = 0$$

In the end get

$$H = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 \, 2\varepsilon_k} \, \varepsilon_k \left[\hat{b}^{+}_{\vec{k},r} \, \hat{b}_{\vec{k},r} - \hat{d}_{\vec{k},r} \, \hat{d}^{+}_{\vec{k},r} \right].$$

Still not positive definite? Really, if we define some commutation relation for $\hat{d}, \hat{d}^{+}$ $\Rightarrow$ would get $\hat{b}^{+}\hat{b} - \hat{d}^{+}\hat{d}$ ~ not good!

Define anti-commutation relations:

$$\{\hat{b}_{\vec{k},r}, \hat{b}^{+}_{\vec{k}',r'}\} = \{\hat{d}_{\vec{k},r}, \hat{d}^{+}_{\vec{k}',r'}\} = (2\pi)^3\, 2\mathcal{E}_k\, \delta_{rr'}\, \delta^3(\vec{k}-\vec{k}')$$

$$\{\hat{b}_{\vec{k},r}, \hat{b}_{\vec{k}',r'}\} = \{\hat{b}^{+}_{\vec{k},r}, \hat{b}^{+}_{\vec{k}',r'}\} = 0$$

$$\{\hat{d}_{\vec{k},r}, \hat{d}_{\vec{k}',r'}\} = \{\hat{d}^{+}_{\vec{k},r}, \hat{d}^{+}_{\vec{k}',r'}\} = 0$$

$$\{\hat{b}, \hat{d}^{+}\} = 0$$
$$\{d, \hat{b}^{+}\} = 0$$
$$\{\hat{b}, \hat{d}\} = \{\hat{b}^{+}, \hat{d}^{+}\} = 0$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ = anti-commutators

$\Rightarrow$ dropping ∞ number get

$$\boxed{H = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k}\, \mathcal{E}_k \left[\hat{b}^{+}_{\vec{k},r}\hat{b}_{\vec{k},r} + \hat{d}^{+}_{\vec{k},r} d_{\vec{k},r}\right]}$$

Now it's positive-definite!

For the fields get $\{\psi_\alpha(\vec{x},t), \underbrace{\pi_\beta(\vec{x}',t)}_{= i\psi_\beta^{+}}\} = i\delta_{\alpha\beta}\delta(\vec{x}-\vec{y})$

$$\{\psi_\alpha, \psi_\beta\} = \{\psi^{+}_\alpha, \psi^{+}_\beta\} = 0$$

anti-commutation relations.

$\Rightarrow$ all operators anti-commute.

Time evolution:

$$+i\frac{\partial}{\partial t}\psi(x) = [\psi, H]$$
$$i\frac{\partial}{\partial t}\bar{\psi}(x) = [\bar{\psi}, H]$$

Still uses commutators (can show)

see HW