Last time We quantized the free Dirac field:

solved Dirac equation, $[i\not{\partial} - m]\psi(x) = 0$.

$$\boxed{\psi(x) = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2E_k} \left\{ \hat{b}_{\vec{k},r}\, u_r(\vec{k})\, e^{-ik\cdot x} + \hat{d}^{+}_{\vec{k},r}\, v_r(\vec{k}) e^{ik\cdot x} \right\}.}$$

Showed that the Dirac Hamiltonian becomes

$$\boxed{H = \sum_{r=1,2} \int \frac{d^3k}{(2\pi)^3 2E_k} E_k \left[\hat{b}^{+}_{\vec{k},r} \hat{b}_{\vec{k},r} + \hat{d}^{+}_{\vec{k},r} \hat{d}_{\vec{k},r} \right]}$$

if we impose anti-commutation relations

$$\boxed{\{ \hat{b}_{\vec{k},r}, \hat{b}^{+}_{\vec{k}',r'} \} = \{ \hat{d}_{\vec{k},r}, \hat{d}^{+}_{\vec{k}',r'} \} = (2\pi)^3\, 2E_k\, \delta_{rr'}\, \delta^3(\vec{k}-\vec{k}')}$$

(all other anti-commutators are zero.)

For ψ & $\pi \overset{=\, i\psi^{+}}{}$ we have

$$\boxed{\begin{aligned} &\{ \psi_\alpha(\vec{x},t), \pi_\beta(\vec{y},t) \} = i\,\delta_{\alpha\beta}\, \delta^3(\vec{x}-\vec{y}) \\ &\{ \psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t) \} = 0 = \{ \pi_\alpha(\vec{x},t), \pi_\beta(\vec{y},t) \} \end{aligned}}$$

Time evolution: $i\partial_t \psi = [\psi, H]$

Useful formulas: $\bar{u}_r(\vec{k})\, u_s(\vec{k}) = 2m\,\delta_{rs}$

$$\bar{v}_r(\vec{k})\, v_s(\vec{k}) = -2m\,\delta_{rs}$$

$$u_r^{+}(\vec{k})\, u_s(\vec{k}) = 2\,\mathcal{E}_k\,\delta_{rs}$$

$$v_r^{+}(\vec{k})\, v_s(\vec{k}) = 2\,\mathcal{E}_k\,\delta_{rs}$$

$$\left.\begin{aligned} \sum_{r=1}^{2} u_{r,\alpha}(\vec{k})\, \bar{u}_{r,\beta}(\vec{k}) &= \left(\gamma\cdot k + m\right)_{\alpha\beta} \\ \sum_{r=1}^{2} v_r(\vec{k})\, \bar{v}_r(\vec{k}) &= \gamma\cdot k - m. \end{aligned}\right\}\ \text{can prove}$$

Gauge Fields (photons)

$A^\mu = (\Phi, \vec{A}) \sim$ 4-vector $\Rightarrow$ one can build a gauge invariant tensor: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (field strength tensor) $\Rightarrow$ the Lagrangian is $\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right).$

EOM: $$\frac{\delta \mathcal{L}}{\delta A_\mu} - \partial_\nu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)}\right) = 0$$

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = 0\,, \quad \frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} = -\frac{1}{4}\frac{\delta\left(F_{\alpha\beta}F^{\alpha\beta}\right)}{\delta(\partial_\nu A_\mu)} = -\frac{1}{4}\frac{\delta\left(\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right) g^{\alpha\sigma} g^{\beta\rho}\right)}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu}$$

Conserved current of Dirac Lagrangian:

$j^\mu = \bar{\psi}\gamma^\mu\psi$ $\quad$ $\partial_\mu j^\mu = 0$

Conserved charge is

$$Q = \int d^3x \, j^0(\vec{x},t) = \int d^3x \, \psi^+\psi \overset{\text{can show}}{=} \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2E_k} \cdot$$

$$\cdot \left[\hat{b}^+_{\vec{k},r} \hat{b}_{\vec{k},r} - \hat{d}^+_{\vec{k},r} \hat{d}_{\vec{k},r} \right]$$

$\Rightarrow$ just like for complex scalar fields, we see that

$\hat{b}^+_{\vec{k},r} \sim$ creates particles (charge $+1$)

$\hat{d}^+_{\vec{k},r} \sim$ creates anti-particles (charge -1).

both particles & anti-particles could be helicity $+$ or $-$ $\Rightarrow$ 2 d.o.f. each ($r = \pm 1$).

Fock states: $\hat{b}^+_{\vec{k},r} \hat{d}^+_{\vec{p},r'} |0\rangle$, ...

Note: can not have 2 identical particles:

$$\hat{b}^+_{\vec{k},r} \hat{b}^+_{\vec{k},r} |0\rangle = 0 \quad \text{as} \left(\hat{b}^+_{\vec{k},r}\right)^2 = 0$$

(anti-commutation relations)

$\Rightarrow$ Pauli exclusion principle works!

$$\partial_\nu F^{\nu\mu} = j^\mu \Rightarrow \partial_\nu (\partial^\nu A^\mu - \partial^\mu A^\nu) = j^\mu$$

$$\Rightarrow \Box A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu$$

Canonical Quantization of free vector field ($j^\mu = 0$)

$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$

Two ways to do canonical quantization:

(II) in Lorenz gauge $\partial_\mu A^\mu = 0 \Rightarrow \Box A^\mu = 0$

Now, Lorenz gauge has gauge sub-freedom:

$A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda \Rightarrow$ want $\partial_\mu A'^\mu = 0$ too

$\Rightarrow$ if $\partial_\mu A^\mu = 0 \Rightarrow$ need $\Box \Lambda = 0 \Rightarrow$ for any such Λ still have $\partial_\mu A'^\mu = 0$.

Choose Λ such that $A^0 = -\partial^0 \Lambda \Rightarrow$

$\Rightarrow A'^0 = 0$, $\vec{\nabla} \cdot \vec{A}' = 0 \Rightarrow$ can work

(I) in Coulomb gauge $A^0 = 0$, $\vec{\nabla} \cdot \vec{A} = 0$
(radiation)

Can one always pick this gauge? $\Lambda = -\frac{1}{\partial^0} A^0$

$\Rightarrow$ plug back into $\Box \cdot \Lambda = 0 \Rightarrow ((\partial_0)^2 - \vec{\nabla}^2)\Lambda = 0 \Rightarrow$

$\Rightarrow -\partial^0 A^0 = \vec{\nabla}^2 \Lambda \Rightarrow \Lambda = -\frac{\partial^0}{\vec{\nabla}^2} A^0 = -\frac{1}{\partial_0} A^0$

$\Rightarrow [(\partial^0)^2 - \vec{\nabla}^2] A^0 = 0 \Rightarrow \Box A^0 = 0 \sim 0^{th}$ component of maxwell equations, $\xRightarrow{\text{in coul. gauge}}$ valid!

(I) Coulomb gauge quantization

$$A^0 = 0 \, , \quad \vec{\nabla} \cdot \vec{A} = 0.$$

$\pi_i = \dfrac{\delta \mathcal{L}}{\delta \dot{A}^i}$ ~ canonical momenta, A^i ~ free fields (A^0 is fixed)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\pi^i = \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = \frac{\delta \mathcal{L}}{\delta(\partial_0 A_i)} = -F^{0i} = E^i \quad \text{(electric field)}$$

see above. $\quad -\dot{A}^i = \dot{A}_i$

$\Rightarrow$ E^i is the canonical momentum!

(conjugate to A_i ~ lower "i")

$\Rightarrow$ canonical commutation relation could be:

$$[\, A_i(\vec{x}, t), \pi^j(\vec{y}, t) \,] \overset{?}{=} i \, \delta_i^{\,j} \, \delta(\vec{x} - \vec{y})$$

but: this would violate gauge condition $\vec{\nabla} \cdot \vec{A} = 0$

as, if we act with $\vec{\nabla}_x$ on it we get

$$[\, \underbrace{-\vec{\nabla} \cdot \vec{A}(\vec{x}, t)}_{=0}, \pi^j(\vec{y}, t) \,] = i \, \delta_i^{\,j} \, \vec{\nabla}_x \, \delta(\vec{x} - \vec{y}) \neq 0$$

$\Rightarrow$ replace $\delta_i^{\,j}$ with $d_i^{\,j}$ such that $d_i^{\,j} \nabla^i = 0$

$\Rightarrow$ $d_i^{\,j} = \delta_i^{\,j} + \dfrac{\partial_i \partial^j}{\vec{\nabla}^2}$ clearly works ($\partial^i = -\partial_i = \nabla^i$)

$\left(\Rightarrow d_{ij} = g_{ij} + \dfrac{\partial_i \partial_j}{\;} = -\delta_{ij} + \dfrac{\partial_i \partial_j}{\;} \right)$

$$\Rightarrow \begin{aligned} &\left[A_i(\vec{x},t),\ \pi^j(\vec{y},t) \right] = i\left(\delta_i{}^j + \frac{\partial_i \partial^j}{\vec{\nabla}^2} \right) \delta(\vec{x}-\vec{y}) \\ &\left[A_i(\vec{x},t),\ A_j(\vec{y},t) \right] = \left[\pi^i(\vec{x},t),\ \pi^j(\vec{y},t) \right] = 0 \end{aligned}$$

correct Coulomb gauge commutation relations.

$\left([A^i, \pi^j] = i\left(g^{ij} + \frac{\partial^i \partial^j}{\vec{\nabla}^2}\right)\delta(\vec{x}-\vec{y}) = -i\left(\delta^{ij} - \frac{\partial^i\partial^j}{\vec{\nabla}^2}\right)\delta(\vec{x}-\vec{y})\right)$.

Maxwell equations $\quad \Box A^\mu = 0 \Rightarrow \Box A^i = 0$

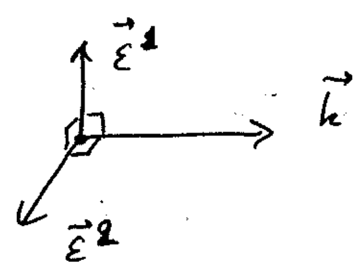
$$\Rightarrow \vec{A}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k} \sum_{\lambda=1}^{2} \vec{\varepsilon}^{\,\lambda}(\vec{k}) \left[\hat{a}_{\vec{k},\lambda}\, e^{-ik\cdot x} + \hat{a}^{+}_{\vec{k},\lambda}\, e^{ik\cdot x} \right]$$

Now $\mathcal{E}_k = |\vec{k}|$ as there is no mass.

$\vec{\varepsilon}^{\,\lambda}(\vec{k})$ ~ polarization vectors, $\lambda = 1, 2$.

Impose gauge condition $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow$

$\Rightarrow \vec{k}\cdot\vec{\varepsilon}^{\,\lambda}(\vec{k}) = 0 \quad \Rightarrow$



choose an orthonormal basis for $\vec{\varepsilon}^{\,\lambda}$'s as shown here ↗

$$\vec{\varepsilon}^{\,\lambda}(\vec{k})\cdot\vec{\varepsilon}^{\,\lambda'}(\vec{k}) = \delta^{\lambda\lambda'}$$

$$\begin{aligned} &\left[\hat{a}_{\vec{k},\lambda},\ \hat{a}^{+}_{\vec{k}',\lambda'} \right] = (2\pi)^3\, 2\mathcal{E}_k\, \delta_{\lambda\lambda'}\, \delta^3(\vec{k}-\vec{k}') \\ &\left[\hat{a}_{\vec{k},\lambda},\ \hat{a}_{\vec{k}',\lambda'} \right] = \left[\hat{a}^{+}_{\vec{k},\lambda},\ \hat{a}^{+}_{\vec{k}',\lambda'} \right] = 0 \end{aligned}$$

commutation relations for creation & annihilation operators

The Hamiltonian is then

$$H = \int d^3x \, [\underset{\dot{A}_i}{\underset{\shortparallel}{\pi^i}} \dot{A}_i - \mathcal{L}] = \int d^3x \left[(\dot{A}_i)^2 - \left(-\tfrac{1}{4}\right) \left[2 \underbrace{F_{0i}}_{\dot{A}_i} \underbrace{F^{0i}}_{-\dot{A}_i} \right.\right.$$

$$\left.\left. + F_{ij} F^{ij} \right]\right] = \int d^3x \left[(\dot{A}_i)^2 - \frac{1}{2} (\dot{A}_i)^2 + \frac{1}{4} (F_{ij})^2 \right]$$

$$= \int d^3x \left[\frac{1}{2} (\dot{A}_i)^2 + \frac{1}{4} \overbrace{(F_{ij})^2}^{2 \vec{B}^2} \right] \quad , \; \vec{B} \sim \text{magnetic field}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

$$\Rightarrow \quad H = \frac{1}{2} \int d^3x \left[\vec{E}^2 + \vec{B}^2 \right] \quad \Rightarrow \text{energy} \geq 0$$

(good.)

Plugging in the $\vec{A}$ - field we get (after some math)

$$\boxed{H = \int \frac{d^3k}{(2\pi)^3 \, 2 \varepsilon_k} \sum_{\lambda=1}^{2} \varepsilon_k \, \hat{a}^{+}_{\vec{k},\lambda} \, \hat{a}_{\vec{k},\lambda}}$$

$\hat{a}^{+}_{\vec{k},\lambda}$ ~ creation operator

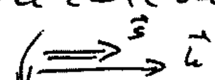
$\hat{a}_{\vec{k},\lambda}$ ~ annihilation operator

$\lambda = 1, 2$ $\Rightarrow$ photons have 2 degrees of freedom (2 polarizations)

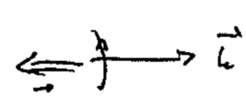
$\vec{\varepsilon}^{\,1}, \vec{\varepsilon}^{\,2}$ $\Rightarrow$ can choose circular polarizations

$$\vec{\varepsilon}^{\,\pm} = \frac{\vec{\varepsilon}^{\,1} \pm i \vec{\varepsilon}^{\,2}}{\sqrt{2}}$$

$+$ ~ helicity $+1$



$-$ ~ helicity -1.



(II) Lorenz gauge quantization

in Coulomb gauge physics is still Lorentz-invariant, but not manifestly so.

$\Rightarrow$ let's work in Lorenz gauge $\partial_\mu A^\mu = 0$

$A^\mu(x)$ ~ our fields

$$\pi^\mu = \frac{\delta \mathcal{L}}{\delta \dot{A}_\mu} = \frac{\delta \mathcal{L}}{\delta(\partial_0 A_\mu)} = F^{\mu 0} = -F^{0\mu}$$ ~ canonical momenta.

Problem: $\pi^0 = -F^{00} = \emptyset$. No π^0? Bad, since would not have commutation relation

$[A_0(\vec{x},t), \pi_0(\vec{y},t)] = i g_{00}\, \delta(\vec{x}-\vec{y})$. ...

Modify the Lagrangian to

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2$$

(not gauge invariant)

$$\Rightarrow \frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} = \underset{\text{old term}}{F^{\mu\nu}} - \underset{\text{gauge-fixing term}}{\frac{\lambda}{2} \cdot \frac{\delta(g^{\alpha\beta}\partial_\alpha A_\beta\, g^{\sigma\rho}\partial_\sigma A_\rho)}{\delta(\partial_\nu A_\mu)}} =$$

$$= F^{\mu\nu} - \lambda \underbrace{\underbrace{g^{\alpha\beta} g_\alpha{}^\nu}_{g^{\beta\nu}} g_\beta{}^\mu}_{g^{\mu\nu}} (\partial_\rho A^\rho) = F^{\mu\nu} - \lambda g^{\mu\nu}(\partial_\rho A^\rho)$$

$\frac{\delta \mathcal{L}}{\delta A_\mu} = 0$ still.

$$\partial_\nu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} \right] = 0 \Rightarrow \partial_\nu F^{\mu\nu} - \lambda \partial^\mu \partial_\rho A^\rho = 0$$

$$\Rightarrow \partial_\nu \partial^\mu A^\nu - \Box A^\mu - \lambda \partial^\mu \partial_\rho A^\rho = 0$$

$$\Box A^\mu - (1-\lambda) \partial^\mu \partial_\nu A^\nu = 0$$

$\Rightarrow \Box \partial_\mu A^\mu = 0 \Rightarrow$ classically can set boundary conditions to have $\partial_\mu A^\mu = 0$ and $\Box A^\mu = 0$.

NOT SO in QM

$\lambda = 1 \Rightarrow$ "Feynman gauge" $\Rightarrow$ get $\Box A^\mu = 0$ back.

$\Rightarrow$ the same classical physics with a different Lagrangian: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2$.

Now, $\pi^0 = \dfrac{\delta \mathcal{L}}{\delta \dot{A}_0} = -\partial_\mu A^\mu$. No longer need to impose $\partial_\mu A^\mu = 0$ explicitly $\Rightarrow \pi^0$ does not have to be $\emptyset$.

Quantize the system:

$$[A_\mu(\vec{x},t), \pi_\nu(\vec{y},t)] = i g_{\mu\nu} \delta(\vec{x}-\vec{y})$$

$$[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = [\pi_\mu(\vec{x},t), \pi_\nu(\vec{y},t)] = 0$$

$$A_\mu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \sum_{\lambda=0}^{3} \varepsilon_\mu^\lambda(\vec{k}) \left[\hat{a}_{\vec{k},\lambda} e^{-ik\cdot x} + \hat{a}^\dagger_{\vec{k},\lambda} e^{ik\cdot x} \right]$$

Choose the basis of polarization vectors in such a way that $\varepsilon^\lambda \cdot \varepsilon^{\lambda'} = g^{\lambda\lambda'}$

One then gets:

$$\boxed{\left[\hat{a}_{\vec{k},\lambda}\,,\ \hat{a}^{+}_{\vec{k}',\lambda'}\right] = -\,g_{\lambda\lambda'}\,(2\pi)^3\, 2\varepsilon_K\,\delta(\vec{k}-\vec{k}')}$$

all other commutators are zero.

if $k^\mu = (k, 0, 0, k) \Rightarrow$

$$\varepsilon^0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \varepsilon^1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \varepsilon^2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \varepsilon^3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$\varepsilon^{1,2}\cdot k = 0$ (transverse)

$\varepsilon^3 \sim$ longitudinal, $\varepsilon^0 \sim$ time-like.

Fock space: $\hat{a}^{+}_{\vec{k},\lambda}|0\rangle, \ldots$

$$|1,\lambda\rangle = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_K} f(\vec{k})\,\hat{a}^{+}_{\vec{k},\lambda}|0\rangle \quad \sim \text{one-photon state}$$

<u>Problem</u>: $\langle 1, \lambda=0 | 1, \lambda=0\rangle = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_K}\,\frac{d^3k'}{(2\pi)^3 2\varepsilon_{K'}}\cdot f(\vec{k})\,f^*(\vec{k}')$

$$\cdot\langle 0|\underbrace{\hat{a}_{\vec{k}',\lambda=0}\ \hat{a}^{+}_{\vec{k},0}}|0\rangle = -\int \frac{d^3k}{(2\pi)^3 2\varepsilon_K}|f(\vec{k})|^2\langle 0|0\rangle$$

$$\left[\hat{a}_{\vec{k}',0}\,,\ \hat{a}^{+}_{\vec{k},0}\right] = -(2\pi)^3\, 2\varepsilon_K\,\delta(\vec{k}-\vec{k}')$$

$\Rightarrow$ negative norm states!

$$\boxed{H = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_K}\,\varepsilon_K\left[\sum_{\lambda=1}^{3}\hat{a}^{+}_{\vec{k},\lambda}\,\hat{a}_{\vec{k},\lambda} - \hat{a}^{+}_{\vec{k},0}\,\hat{a}_{\vec{k}0}\right]} \sim \text{negative energy problem} \ldots$$