Last time | Canonical Quantization of Free Vector Field (cont'd)

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Lorenz gauge $\boxed{\partial_\mu A^\mu = 0}$.

if A_μ are 4 fields $\Rightarrow \pi^\mu = F^{\mu 0} \Rightarrow \pi^0 = 0$

$\Rightarrow$ problem

Use $\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} (\partial_\mu A^\mu)^2}$

$$\pi^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_0 A_\mu)} = F^{\mu 0} - \lambda g^{\mu 0} (\partial_\rho A^\rho)$$

$$\Rightarrow \pi^0 = -\lambda \, \partial_\rho A^\rho \neq 0$$

$$\boxed{\begin{aligned} &[A_\mu(\vec{x},t), \pi_\nu(\vec{y},t)] = i g_{\mu\nu} \delta^3(\vec{x}-\vec{y}) \\ &[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = 0 = [\pi_\mu(\vec{x},t), \pi_\nu(\vec{y},t)] \end{aligned}}$$

$$\boxed{A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_{\lambda=0}^{3} \varepsilon_\mu^\lambda(k) \left[\hat{a}_{\vec{k},\lambda} e^{-ik\cdot x} + \hat{a}^\dagger_{\vec{k},\lambda} e^{ik\cdot x} \right]}$$

real polarization vector $\varepsilon^{\lambda} \cdot \varepsilon^{\lambda'} = g^{\lambda\lambda'}$

$$\left[\hat{a}_{\vec{k},\lambda} , \hat{a}^{+}_{\vec{k}',\lambda'} \right] = - g_{\lambda\lambda'} (2\pi)^3 \, 2E_k \, \delta^3(\vec{k} - \vec{k}')$$

$$\partial_\nu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} \right] = 0 \Rightarrow \partial_\nu F^{\mu\nu} - \lambda \, \partial^\mu \partial_\rho A^\rho = 0$$

$$\Rightarrow \partial_\nu \partial^\mu A^\nu - \Box A^\mu - \lambda \, \partial^\mu \partial_\rho A^\rho = 0$$

$$\Box A^\mu - (1-\lambda)\, \partial^\mu \partial_\nu A^\nu = 0 \Rightarrow$$

$\Box \partial_\mu A^\mu = 0 \Rightarrow$ NOT SO in QM

$\Rightarrow$ classically can set boundary conditions to have $\partial_\mu A^\mu = 0$ and $\Box A^\mu = 0$.

$\lambda = 1 \Rightarrow$ "Feynman gauge" $\Rightarrow$ get $\Box A^\mu = 0$ back.

$\Rightarrow$ the same classical physics with a different

Lagrangian: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2$.

Now, $\pi^0 = \dfrac{\delta \mathcal{L}}{\delta \dot{A}_0} = -\partial_\mu A^\mu$. No longer need to impose $\partial_\mu A^\mu = 0$ explicitly $\Rightarrow$ π^0 does not have to be $\emptyset$.

Quantize the system:

$$[A_\mu(\vec{x},t), \pi_\nu(\vec{y},t)] = i g_{\mu\nu} \, \delta(\vec{x}-\vec{y})$$

$$[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = [\pi_\mu(\vec{x},t), \pi_\nu(\vec{y},t)] = 0$$

$$A_\mu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_{\lambda=0}^{3} \varepsilon_\mu^\lambda(\vec{k}) \left[\hat{a}_{\vec{k},\lambda} e^{-ik\cdot x} + \hat{a}^+_{\vec{k},\lambda} e^{ik\cdot x} \right]$$

choose the basis of polarization vectors in such a way that $\varepsilon^\lambda \cdot \varepsilon^{\lambda'} = g^{\lambda\lambda'}$

One then gets:

$$\left[\hat{a}_{\vec{k},\lambda}, \hat{a}^{+}_{\vec{k}',\lambda'}\right] = -g_{\lambda\lambda'}(2\pi)^3\, 2\varepsilon_k\, \delta(\vec{k}-\vec{k}')$$

all other commutators are zero.

if $k^\mu = (k, 0, 0, k) \Rightarrow$

$$\varepsilon^0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \varepsilon^1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \varepsilon^2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \varepsilon^3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$\varepsilon^{1,2} \cdot k = 0$ (transverse)

$\varepsilon^3 \sim$ longitudinal, $\varepsilon^0 \sim$ time-like.

Fock space: $\hat{a}^{+}_{\vec{k},\lambda}|0\rangle, \ldots$

$$|1,\lambda\rangle = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} f(\vec{k})\, \hat{a}^{+}_{\vec{k},\lambda}|0\rangle$$ $\sim$ one-photon state

Problem: $$\langle 1, \lambda=0 | 1, \lambda=0\rangle = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}} \cdot f(\vec{k}) f^*(\vec{k}')$$

$$\cdot \langle 0| \underbrace{\hat{a}_{\vec{k}',\lambda=0}\, \hat{a}^{+}_{\vec{k},0}}_{} |0\rangle = -\int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} |f(\vec{k})|^2 \langle 0|0\rangle$$

$$" \quad \left[\hat{a}_{\vec{k}',0}, \hat{a}^{+}_{\vec{k},0}\right] = -(2\pi)^3\, 2\varepsilon_k\, \delta(\vec{k}-\vec{k}')$$

$\Rightarrow$ negative norm states!

$$H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\, \varepsilon_k \left[\sum_{\lambda=1}^{3} \hat{a}^{+}_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda} - \hat{a}^{+}_{\vec{k},0}\, \hat{a}_{\vec{k},0}\right]$$ $\sim$ negative energy problem...

$\Rightarrow$ way out: demand that while $\partial_\mu A^\mu \neq 0$ at the operator level, for physical states one has

$$\partial_\mu A^{(+)\mu} | \psi \rangle = 0$$

where (+) denotes positive-energy part of A_μ (the term with $\hat{a}_{\vec{k},\lambda}$).

$$\langle \psi | \partial_\mu A^\mu | \psi \rangle = \langle \psi | \partial_\mu A^{(+)\mu} + \partial_\mu A^{(-)\mu} | \psi \rangle = 0.$$

(Gupta & Bleuler method). $\langle \psi | \partial_\mu A^{(-)\mu} = \left[\partial_\mu A^{(+)\mu} | \psi \rangle \right]^+ \Rightarrow$

$$\Rightarrow \partial_\mu A^{(+)\mu} | \psi \rangle = 0 \Rightarrow \sum_{\lambda=0}^{3} k^\mu \varepsilon_\mu^\lambda(\vec{k})\, \hat{a}_{\vec{k},\lambda} | \psi \rangle = 0$$

$$\Rightarrow \left(k^\mu \varepsilon_\mu^0\, \hat{a}_{\vec{k},0} + k^\mu \varepsilon_\mu^3\, \hat{a}_{\vec{k},3} \right) | \psi \rangle = 0$$

as $\varepsilon^{1,2} \cdot k = 0$. Now $k \cdot \varepsilon^0 = k^0 = - k \cdot \varepsilon^3$

$$\Rightarrow \left(\hat{a}_{\vec{k},0} - \hat{a}_{\vec{k},3} \right) | \psi \rangle = 0$$

$\Rightarrow$ physical states are mixtures of longitudinal & time-like photons (along with the transverse photons).

$$\Rightarrow \langle \psi | \hat{a}^+_{\vec{k},0}\, \hat{a}_{\vec{k},0} | \psi \rangle = \langle \psi | \hat{a}^+_{\vec{k},3}\, \hat{a}_{\vec{k},3} | \psi \rangle$$

$\Rightarrow$ only transverse photons contribute in H:

$$\langle \psi | H | \psi \rangle = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \varepsilon_k \cdot \langle \psi | \sum_{\lambda=1}^{2} \hat{a}^+_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda} | \psi \rangle \geqslant 0$$

$\Rightarrow$ no negative energy problem any more.

Massive Vector Field

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu$$
, m is the mass

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$
Proca eqn's.

$\partial_\mu A^\mu = 0$ always $\Rightarrow$ only A^i are independent d.o.f. $\Rightarrow$

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \sum_{\lambda=1}^{3} \varepsilon_\mu^{(\lambda)}(\vec{k}) \left[\hat{a}_{\vec{k},\lambda} e^{-ik\cdot x} + \hat{a}^+_{\vec{k},\lambda} e^{ik\cdot x} \right]$$

with $\left[\hat{a}_{\vec{k},\lambda}, \hat{a}^+_{\vec{k}',\lambda'} \right] = \delta_{\lambda\lambda'} (2\pi)^3 2\varepsilon_k \delta(\vec{k}-\vec{k}')$

$k \cdot \varepsilon^{(\lambda)} = 0$ for $\lambda = 1, 2, 3$ $\Rightarrow$ in rest frame

have $k^\mu = (m, 0, 0, 0)$ $\Rightarrow$

$\varepsilon_\mu^{(1)} = (0, 1, 0, 0)$, $\varepsilon_\mu^{(2)} = (0, 0, 1, 0)$, $\varepsilon_\mu^{(3)} = (0, 0, 0, 1)$.

3 degrees of freedom (all physical)

$$H = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \varepsilon_k \sum_{\lambda=1}^{3} \hat{a}^+_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} .$$

Time evolution: $-i\partial_t A_\mu = [H, A_\mu]$ for both massive & massless vector fields.

(Note that A^0 is not a dof as it is eliminated using $\partial_\mu A^\mu = 0$. No worries about $\pi^0 = 0$ again.)