Last time

Quarks (cont'd)

I_3 ~ isospin projection

B ~ baryon number

S ~ strangeness

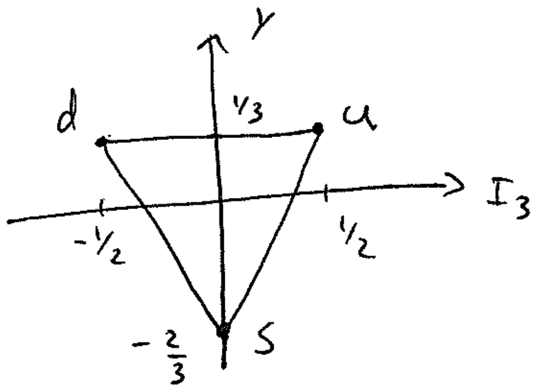
$$\boxed{Q = I_3 + \frac{B+S}{2} = I_3 + \frac{Y}{2}}$$ Gell-Mann - - Nishijima f-la

Y ~ hypercharge

u - quark: $I_3 = +\frac{1}{2}$, $B = \frac{1}{3}$, $S = 0 \Rightarrow Y = \frac{1}{3}$

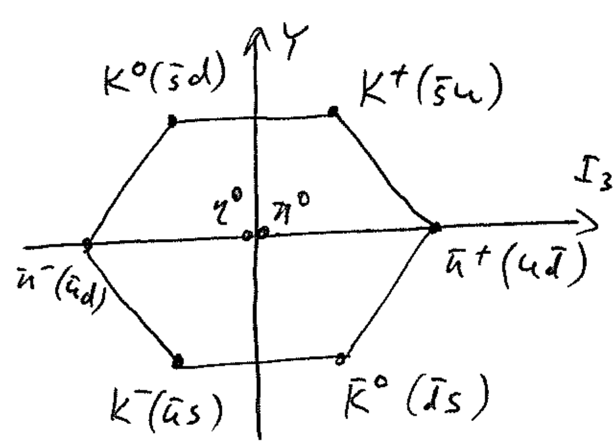
d - quark: $I_3 = -\frac{1}{2}$, $B = \frac{1}{3}$, $S = 0 \Rightarrow Y = \frac{1}{3}$

s - quark : $I_3 = 0$, $B = \frac{1}{3}$, $S = -1 \Rightarrow Y = -\frac{2}{3}$

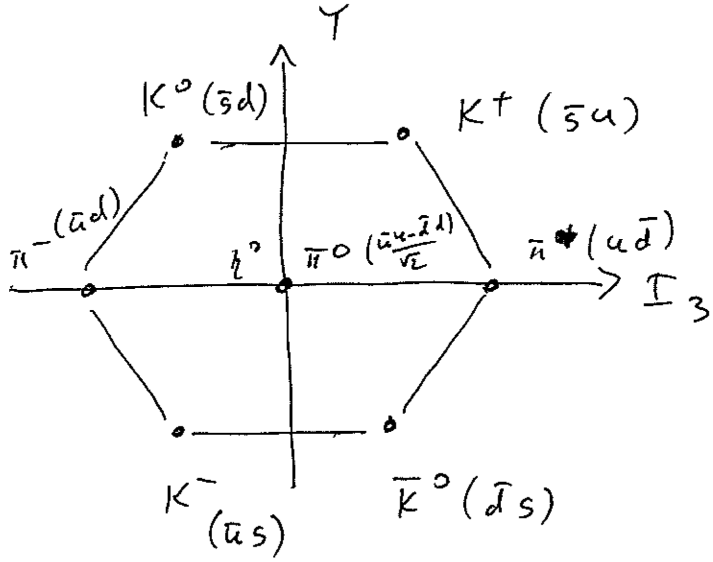


$\Rightarrow$

0^- mesons



$$\pi^0 = \frac{\bar{u}u - \bar{d}d}{\sqrt{2}}, \quad \eta^0 = \frac{\bar{u}u + \bar{d}d - 2\bar{s}s}{\sqrt{6}}.$$



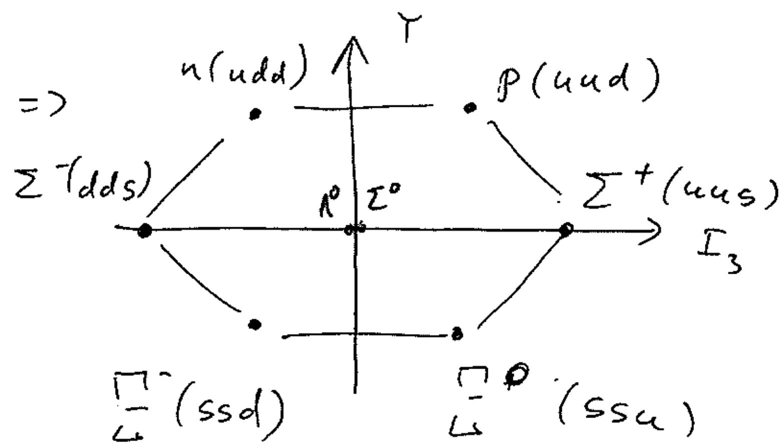
$K^0 = \bar{s}d \Rightarrow Y = 1,$

$I_3 = \frac{1}{2}$, etc.

baryons: $p \sim uud$, $n \sim udd$, $\Sigma^+ = uus$, $\Sigma^- = dds$,

$\Sigma^0 = s\,\frac{ud + du}{\sqrt{2}}$, $\Xi^0 \sim ssu$, $\Xi^- = ssd$, $\Lambda^0 \sim s\,\frac{ud - du}{\sqrt{2}}$

$\Rightarrow$



e.g. $p\,(uud) \Rightarrow$

$Y = B + S = +\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = +1$

$I_3 = +\frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}.$

etc.

$\Rightarrow$ in reality there are 6 quark flavors:

u, d, s, c, b, t
c ← charm, b ← bottom, t ← top

$c, b, t \sim$ heaviest, do not contribute to the lightest hadrons considered so far.

$\frac{3}{2}^+$ baryons form a decuplet:

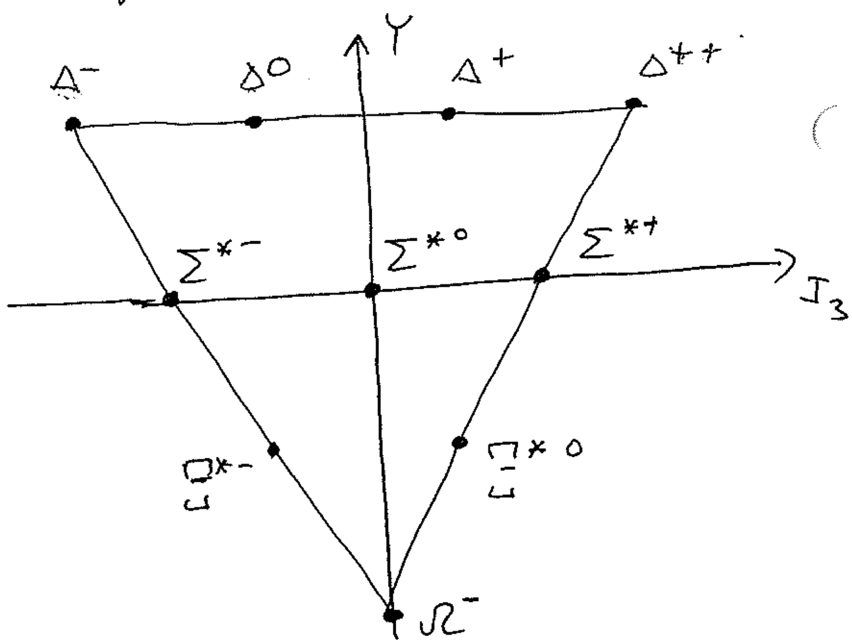
$\Delta^{++} = uuu$, $\Delta^+ \sim uud$,

$\Delta^0 \sim udd$, $\Delta^- \sim ddd$

$\Sigma^{*+} \sim suu$, $\Sigma^{*0} \sim sud$,

$\Sigma^{*-} \sim sdd$, $\Xi^{*0} \sim ssu$,

$\Xi^{*-} \sim ssd$, $\Omega^- \sim sss$.



$\Rightarrow$ seems OK? but let's look at Δ^{++} for instance: it has spin-$3/2$ $\Rightarrow$ the spin state is

$\underset{u\,u\,u}{|\uparrow\uparrow\uparrow\rangle} = |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle$ $\Rightarrow$ symmetric

the isospin state is $|\uparrow\uparrow\uparrow\rangle$ too ~ also symmetric!

What happens to Pauli principle ~ the full wave function $\Psi = \Psi_{spin} \otimes \Psi_{isospin}$ has to be anti-symmetric! (Fermi-Dirac statistic)

$\Psi_{spatial}$ is symmetric too (ground state for uuu)

$\Rightarrow$ the way out is to postulate a new quantum number called color (Greenberg, Han, Nambu '64-'66)

~~there~~ are 3 colors: $i = 1, 2, 3$ $\Rightarrow$ $u_i(x)$ ~ up quark w.f.

$\Rightarrow \Delta^{++} \propto \varepsilon^{ijk} \, u_i(x_1) \, u_j(x_2) \, u_k(x_3)$

anti-symmetric.

$\Rightarrow$ let us summarize our knowledge about quarks in a table:

	Q	B	I	I_3	S	mass
u (up)	$+2/3$	$1/3$	$\frac{1}{2}$	$\frac{1}{2}$	0	1,8 − 3,0 MeV
d (down)	$-1/3$	$1/3$	$\frac{1}{2}$	$-\frac{1}{2}$	0	4,5 − 5,3 MeV
s (strange)	$-1/3$	$1/3$	×	×	−1	~ 95 MeV
c (charm)	$+2/3$	$1/3$	×	×	0	~ 1,27 GeV
b (bottom)	$-1/3$	$1/3$	×	×	0	~ 4,2 GeV
t (top)	$+2/3$	$1/3$	×	×	0	~ 173 GeV

All quantum numbers flip signs for anti-quarks: $\bar{u}, \bar{d}, \bar{s}, \bar{c}, \bar{b}, \bar{t}$. (Baryon # of $\bar{u}$ is $-1/3$, e.g.)

All quarks are fermions $\Rightarrow$ should be described by Dirac spinors $q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$
(not their own anti-particles $\Rightarrow$ not Majorana)

$\Rightarrow q_\alpha, \; \alpha = 1,2,3,4$ would be correct.

But: what about different colors & flavors?

... index, $i = 1, 2, 3$

$\Rightarrow$ write q_{α}^{if} — flavor index $f = u, d, s, c, b, t$; spinor index α

$\Rightarrow$ quark Lagrangian is

$$\boxed{\mathcal{L}_{\text{quarks}} = \bar{q}^{if} \left(i \gamma \cdot \partial - m_f \right) q^{if}}$$

~ sum over i, f implied, m_f ~ (bare) quark masses

$\Rightarrow$ is that it for strong interactions?

No, quarks should be able to interact with each other!

$\Rightarrow$ one needs gluons: $A_\mu^a(x)$, $a = 1, \ldots, 8$ ~ 8 different gauge fields

$$\boxed{\mathcal{L}_{\text{gluons}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}$$

with $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$

g ~ gluon self-coupling constant, f^{abc} ~ structure constants of $SU(3)$

$\Rightarrow$ What about quark-gluon interactions?

$$\boxed{\mathcal{L}_{\text{int}} = g\, \bar{q}^{if} \gamma^\mu A_\mu^a \left(t^a \right)_{ij} q^{jf}}$$

, where $(t^a)_{ij}$ are 3×3 matrices (generators of $SU(3)$), $a = 1, \ldots, 8$, $i, j = 1, 2, 3$

$\Rightarrow$ putting all this together write the Lagrangian for Quantum Chromodynamics (QCD) – the theory of strong interactions:

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^{if} (i\gamma \cdot \partial - m_f) q^{if} - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + g\, \bar{q}^{if} \gamma^\mu A^a_\mu (t^a)_{ij}\, q^{jf}}$$

Elements of Group Theory

Def. A Group G is a set of elements with a multiplication law having the following properties:

(i) Closure: if $f, g \in G \Rightarrow h = f \cdot g \in G$

(ii) Associativity: $f, g, h \in G \Rightarrow f \cdot (g \cdot h) = (f \cdot g) \cdot h$

(iii) Identity: $\exists\, e \in G \;\; \forall f \in G : ef = fe = f$

(iv) Inverse element: $\forall f \in G \;\; \exists f^{-1} \in G : f f^{-1} = f^{-1} f = e$.

Example: $\{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3}{2}\pi}\}$ form a group (why?). $\mathbb{Z}_4$ " $\{1, i, -1, -i\}$.

Integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ form a group. What is e there? (what is "multiplication" law?)

Def. $H \subset G \Rightarrow H$ is a subgroup.

Def. A group is called Abelian if for any $f, g \in G$: $f \cdot g = g \cdot f$

otherwise it is called non-Abelian ($f \cdot g \neq g \cdot f$)

Example (important!) $n \times n$ unitary matrices form a group: $UU^+ = U^+U = \mathbb{1}$ (unitary matrices).

Def. Such group is denoted $U(n)$, ($e = \mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$) and is called the unitary group.

Sub-example $U(1)$: 1×1 matrices $\Rightarrow$ $e^{i\varphi}$, $\varphi \sim \#$

$\varphi \in \mathbb{R}$ — form a group, $e = 1$.

Def. $n \times n$ unitary matrices with unit determinant ($UU^+ = U^+U = \mathbb{1}$, $\det U = +1$) form a group too!

It is called special unitary group and is denoted

$SU(n)$. (Orthogonal matrices $U^TU = UU^T = \mathbb{1}$ with $\det U = +1$ form $SO(n)$, O = orthogonal)

Def. A representation of group G is a mapping D of group elements: $f \in G$: $f \to D(f)$, where $D(f)$ is a space of linear operators (e.g. matrices) such that:

(i) $D(e) = \mathbb{1}$

(ii) $D(g_1)D(g_2) = D(g_1 g_2)$ for $g_1, g_2 \in G$.