Last time | Finished talking about quarks: introduced color quantum #, then defined quark and gluon fields:

$$q^{if}_{\alpha}$$

color $\rightarrow i$, $i=1,2,3$; $f \leftarrow$ flavor u,d,s,c,b,t

α: spinor index, $\alpha = 1,2,3,4$

$$A^{a}_{\mu}$$

$a \leftarrow$ gluon color index, $a = 1, \ldots, 8$

μ: Lorentz index

We wrote the QCD Lagrangian:

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^{if}(i\not\partial - m_f)q^{if} - \frac{1}{4}F^{a}_{\mu\nu}F^{a\mu\nu} + g\,\bar{q}^{if}\gamma^{\mu}A^{a}_{\mu}(t^{a})_{ij}\,q^{jf}}$$

Sum over repeated indices is implied

$$F^{a}_{\mu\nu} = \partial_\mu A^{a}_{\nu} - \partial_\nu A^{a}_{\mu} + g f^{abc} A^{b}_{\mu} A^{c}_{\nu}$$

$\uparrow$ field strength tensor, f^{abc} = real #'s, anti-symmetric

Elements of Group Theory

Group: a set with mult. law and

(i) Closure (iii) Identity

(ii) Associativity (iv) Inverse

$\forall f, g \in G : f \circ g = g \circ f \Rightarrow$ group is Abelian

otherwise non-Abelian

$U(n)$: unitary $n \times n$ matrices $(U^+U = UU^+ = \mathbb{1})$

$SU(n)$: — " — ⊕ $\det U = +1$.

$O(n)$: orthogonal $n \times n$ matrices $(OO^T = O^T O = \mathbb{1})$

$SO(n)$: — " — ⊕ $\det O = +1$.

$U(1) \sim$ Abelian group

$SU(2), SU(3) \sim$ non-Abelian

$\Rightarrow$ putting all this together write the Lagrangian for Quantum Chromodynamics (QCD) – the theory of strong interactions:

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^{if}(i\gamma\cdot\partial - m_f)q^{if} - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + g\,\bar{q}^{if}\gamma^\mu A^a_\mu (t^a)_{ij}\, q^{jf}}$$

Elements of Group Theory

Def. A Group G is a set of elements with a multiplication law having the following properties:

(i) Closure: if $f, g \in G \Rightarrow h = f\cdot g \in G$

(ii) Associativity: $f, g, h \in G \Rightarrow f\cdot(g\cdot h) = (f\cdot g)\cdot h$

(iii) Identity: $\exists\, e \in G \;\; \forall f \in G : ef = fe = f$

(iv) Inverse element: $\forall f \in G \;\exists f^{-1} \in G : ff^{-1} = f^{-1}f = e$.

Example: $\{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3}{2}\pi}\}$ form a group (why?). $\mathbb{Z}_4$ $\quad \overset{\shortparallel}{\{1, i, -1, -i\}}$.

Integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ form a group.

What is e there?
(what is "multiplication" here?)

Def. $H \subset G \Rightarrow H$ is a subgroup.

Def. A group is called Abelian if for any

$f, g \in G$: $f \cdot g = g \cdot f$

otherwise it is called non-Abelian ($f \cdot g \neq g \cdot f$)

<u>Example (important!)</u> $n \times n$ unitary matrices form a group: $U U^+ = U^+ U = \mathbb{1}$ (unitary matrices).

Def. Such group is denoted $U(n)$, $\left(e = \mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right)$

and is called the <u>unitary group</u>.

<u>Sub-example</u> $U(1)$: 1×1 matrices $\Rightarrow$ $e^{i\varphi}$, $\varphi \sim \#$

$\varphi \in \mathbb{R}$ ~ form a group, $e = 1$.

Def. $n \times n$ unitary matrices with unit determinant ($U U^+ = U^+ U = \mathbb{1}$, $\det U = +1$) form a group too!

It is called <u>special unitary group</u> and is denoted

$SU(n)$. (Orthogonal matrices $U^T U = U U^T = \mathbb{1}$ with $\det U = +1$ form $SO(n)$, O = orthogonal)

Def. A <u>representation</u> of group G is a mapping D of group elements : $f \in G$: $f \to D(f)$, where $D(f)$ is a space of linear operators (e.g. matrices) such that:

(i) $D(e) = \mathbb{1}$

(ii) $D(g_1) D(g_2) = D(g_1 g_2)$ for $g_1, g_2 \in G$.

Take a group $\mathbb{Z}_4$: it has $\{e, g_1, g_2, g_3\}$

Our example $\{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}}\} = \{D(e), D(g_1), D(g_2), D(g_3)\}$

is one of the many possible representations of $\mathbb{Z}_4$.

Def. Dimension of representation is the dimension of the space of D-matrices.

Def. Representation is called reducible if

$\exists M$ (matrix) such that

$$M\, D(g)\, M^{-1} = \begin{bmatrix} D_1(g) & 0 \\ 0 & D_2(g) \\ & & \ddots \end{bmatrix} \quad \text{for } \forall g \in G.$$

$\Rightarrow D = D_1 \oplus D_2 \oplus \ldots$

a representation is called irreducible if no such matrix M exists.

Def. For two groups $G = \{g_1, g_2, \ldots\}$, $H = \{h_1, h_2, \ldots\}$ define direct-product group $G \times H = \{g_i h_j\}$

such that $\quad g_k h_\ell \cdot g_m h_n = g_k g_m \cdot h_\ell h_n$.

Lie Groups

Imagine a group G with elements smoothly dependent on a continuous set of parameters α_i, $i = 1, \ldots, N$: $\quad g(\alpha_i) \in G$

$\Rightarrow$ assume that $g(\alpha_i = 0) = e$ (the identity element)

$\Rightarrow$ for a representation of the group:

$$D(\alpha_i = 0) = \mathbb{1}.$$

Taylor expand $D(\alpha_i)$ near 0:

$$D(\delta\alpha_i) = \mathbb{1} + i\,\delta\alpha_i X_i + \ldots = \mathbb{1} + i\,\delta\vec{\alpha}\cdot\vec{X}$$

(summation over repeating indices assumed)

Def. X_i are called <u>generators</u> of the group.

definition too

$\delta\vec{\alpha} = \frac{\vec{\alpha}}{k}$, $k \sim$ integer

$$D(\alpha_i) = D(\delta\alpha_i)\,D(\delta\alpha_i)\ldots D(\delta\alpha_i) = \lim_{k\to\infty}\left(\mathbb{1} + i\,\delta\vec{\alpha}\cdot\vec{X}\right)^k$$

$$= \lim_{k\to\infty}\left(\mathbb{1} + i\,\frac{\vec{\alpha}}{k}\cdot\vec{X}\right)^k = e^{i\vec{\alpha}\cdot\vec{X}}.$$

Def. A group with elements depending smoothly on continuous set of parameters α_i, $i = 1, \ldots, N$, with generators X_i is called a <u>Lie group</u>.

$D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{X}}$ $\Rightarrow$ as D can be a matrix

$\Rightarrow$ $\vec{X}$ can be a matrix; therefore in general $[X_i, X_j]$ does not have to be 0.

$$[X_i, X_j] = X_i X_j - X_j X_i$$

$\Rightarrow$ however $D(\vec{\alpha})\, D(\vec{\beta}) = e^{i\alpha\cdot X}\, e^{i\beta\cdot X}$ is also a group element $\Rightarrow e^{i\vec{\alpha}\cdot\vec{X}}\, e^{i\vec{\beta}\cdot\vec{X}} = e^{i\vec{\delta}\cdot\vec{X}}$

$\Rightarrow$ can show that for this to work we need

$$[X_a, X_b] = i f_{abc} X_c$$ Lie algebra of generators

f_{abc} ~ structure constants of the group

$f_{abc} = - f_{bac}$;

Def. Commutator: $[A,B] = A\cdot B - B\cdot A$.

f_{abc} are <u>real</u> for unitary representations D (for hermitean X_a): $D^+ D = D D^+ = \mathbb{1}$

<u>Example</u> take the group $SU(2)$: unitary 2×2 matrices with det $= +1$ $\left(u u^+ = u^+ u = \mathbb{1},\ \det u = +1 \right)$.
(defining representation)
Using Pauli matrices we can define a representation of $SU(2)$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow D(\vec{\alpha}) = e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}}$, $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ a 3-vector.

rotations around $\frac{\vec{\alpha}}{|\vec{\alpha}|}$ axis by angle $|\vec{\alpha}|$.

as $\sigma_i^+ = \sigma_i$ (hermitean) $\Rightarrow$ any 2×2 unitary matrix with det $= +1$ can be represented as $e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}} = U$.

Check: $U U^+ = e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}} e^{-i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}} = \mathbb{1}$.

$\det U = \det e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}} = \left[\text{as } \det e^A = e^{\operatorname{tr} A} \right] = 1$

as $\operatorname{tr} \sigma_i = 0$. $\underbrace{8}_{\text{comp's}} - \underbrace{4}_{\text{cond's}} = 4 - 1 = 3$ (linearly independent)

$\Rightarrow$ there are $2^2 - 1 = 3$ different $n\times n$ traceless hermitean matrices $\Rightarrow \{\sigma_i\}$ use up all possibilities.

Generators: $J_i = \frac{\sigma_i}{2} \Rightarrow D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{J}}$.

$\Rightarrow SU(2)$ is a Lie group

We know that $[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k \Rightarrow$

$$\Rightarrow [J_i, J_j] = i\,\varepsilon_{ijk}\,J_k$$

$\Rightarrow$ generators of $SU(2)$ form a Lie algebra with structure constants ε_{ijk}

ε_{ijk} : totally anti-symmetric Levi-Civita symbol, $\varepsilon_{123} = 1$, $\varepsilon_{ijk} = -\varepsilon_{jik} = \varepsilon_{jki} \ldots$

$\varepsilon_{112} = 0 \ldots$

<u>Another example:</u> SU(3) : 3×3 unitary matrices with det = +1

Define Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(cf. Pauli matrices)}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

↑ fundamental representation

Normalization convention $\mathrm{tr}[\lambda_i \lambda_j] = 2\,\delta_{ij}$.

There are $3^2 - 1 = 8$ traceless hermitean 3×3 matrices

=> these should work.

Generators of SU(3): $t^a = \frac{\lambda^a}{2}$ =>

=> $[t^a, t^b] = i f^{abc}\, t^c$ with structure constants f^{abc}, which are anti-symmetric under the interchange of any two indeces.

=> SU(3) is a Lie group with the generator algebra given above.

a	b	c	f^{abc}
1	2	3	1
1	4	7	1/2
1	5	6	−1/2
2	4	6	1/2
2	5	7	1/2
3	4	5	1/2
3	6	7	−1/2
4	5	8	$\sqrt{3}/2$
6	7	8	$\sqrt{3}/2$

$f_{112} = 0 \quad \ldots$

all other f^{abc}'s can be obtained from this table.

Casimir operator commutes with all generators: in SU(N):

$$\vec{t}^{\,2} = t_1^2 + t_2^2 + \ldots + t_{N^2-1}^2 = \frac{N^2-1}{2N} \equiv C_F$$

$\Rightarrow$ for SU(2) it is $3/4$

for SU(3) it is $4/3$.

$$D(\vec{A}) = e^{i\vec{A}\cdot\vec{t}}, \quad \text{with } \vec{A} = (A_1, A_2, \ldots, A_8).$$

~ an 8-component vector.

Jacobi Identity and the Adjoint Representation

~ go back to some general Lie group with the generators X_a obeying some Lie algebra

$$[X_a, X_b] = i f_{abc} X_c .$$

One can then easily prove Jacobi identity: