Last time | **Lie Groups**

(Def.) a group with elements depending smoothly on a continuous set of parameters $\vec{\alpha} = (\alpha^1, \dots, \alpha^N)$ with generators X_i, $i = 1, \dots, N$ is called a Lie group.

$$D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{X}}$$

$$\boxed{[X_a, X_b] = i f_{abc} X_c}$$ Lie algebra of generators.

f_{abc} ~ structure constants

Examples | $SU(2)$ $\quad D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}$

$\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$

$\vec{J} = \frac{\vec{\sigma}}{2}$ ~ generators, $[J_i, J_j] = i\varepsilon_{ijk} J_k$

SU(2) structure constants

$SU(3)$: $D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{t}}$, $\vec{\alpha} = (\alpha^1, \dots, \alpha^8)$

$t^a = \frac{\lambda^a}{2}$, $a = 1, \dots, 8$ $\quad [t^a, t^b] = i f^{abc} t^c$

λ^a = Gell-Mann matrices $\quad$ SU(3) structure constants

Another example: $SU(3)$: 3×3 unitary matrices with $\det = +1$

Define Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(cf. Pauli matrices)

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

fundamental representation

Normalization convention $\mathrm{tr}[\lambda_i \lambda_j] = 2\,\delta_{ij}$.

There are $3^2 - 1 = 8$ traceless hermitean 3×3 matrices

$\Rightarrow$ these should work.

Generators of $SU(3)$: $t^a = \dfrac{\lambda^a}{2}$ $\Rightarrow$

$\Rightarrow$ $[t^a, t^b] = i f^{abc}\, t^c$ with structure constants f^{abc}, which are anti-symmetric under the interchange of any two indices.

$\Rightarrow$ $SU(3)$ is a Lie group with the generator algebra given above.

a	b	c	f^{abc}
1	2	3	1
1	4	7	1/2
1	5	6	−1/2
2	4	6	1/2
2	5	7	1/2
3	4	5	1/2
3	6	7	−1/2
4	5	8	$\sqrt{3}/2$
6	7	8	$\sqrt{3}/2$

$f_{112} = 0$ ...

all other f^{abc}'s can be obtained from this table.

Casimir operator commutes with all generators: in SU(N):

$\vec{t}^{\,2} = t_1^2 + t_2^2 + \ldots + t_{N^2-1}^2 = \frac{N^2-1}{2N} \equiv C_F$

$\Rightarrow$ for SU(2) it is $3/4$

for SU(3) it is $4/3$.

$$D(\vec{A}) = e^{i \vec{A} \cdot \vec{t}}, \text{ with } \vec{A} = (A_1, A_2, \ldots, A_8)$$

~ an 8-component vector.

Jacobi Identity and the Adjoint Representation

~ go back to some general Lie group with the generators X_a obeying some Lie algebra

$$[X_a, X_b] = i f_{abc} X_c .$$

One can then easily prove Jacobi identity:

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0.$$

(prove this by using definitions of commutators)

$\Rightarrow$ plug in the commutator of Lie algebra to write

$$f_{bdc} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0$$

this relations are obeyed by structure constants of any Lie group, e.g. $SU(n)$.

Define The generators in the adjoint representation by $(T^a)_{bc} = -i f_{abc}$ $\Rightarrow$ the above relation gives

$$[T^a, T^b] = i f_{abc} T^c$$

$\Rightarrow$ they obey the Lie algebra too!

Def. $D(\vec{A}) = e^{i A^a T^a}$ gives the <u>adjoint</u> representation of Lie group. (irreducible representation) adjoint representation of $SU(2)$:

$$(T^1)_{bc} = -i\,\varepsilon_{1bc} \Rightarrow T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$(T^2)_{bc} = -i\,\varepsilon_{2bc} \Rightarrow T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$(T^3)_{bc} = -i\,\varepsilon_{3bc} \Rightarrow T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

can check that $[T^a, T^b] = i\,\varepsilon^{abc}\,T^c$

e.g. $[T^1, T^2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = i\,T^3,$$

as expected.

Tensor Method for $SU(n)$

Consider representation of $SU(n)$ in terms of $n\times n$ unitary matrices U ($UU^+ = \mathbb{1}$) with $\det U = +1$.

Matrices U can be thought of as linear operators acting on the n-dim vectors $a_i = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in C_n$:

$$a_i \to a'_i = U_{ij}\, a_j .$$

(Def.)

A scalar product $a_i^* b_i = a\cdot b$ is invariant under $SU(n)$ transformations:

$$a_i^* b_i \to a'^*_i\, b'_i = U^*_{ij}\, a_j^*\, U_{ik} b_k = a_j^* \underbrace{U^+_{ji} U_{ik}}_{\delta_{jk}} b_k$$

$$= a_j^* b_j$$

(Def.)

Introduce upper indices: $a^i = a_i^*$, $U_i{}^j \equiv U_{ij}$

$U^i{}_j \equiv U^*_{ij}$

$$\Rightarrow \quad a_i \to a'_i = U_i{}^j a_j$$

$$a^i \to a'^i = U^i{}_j\, a^j$$

$\Rightarrow$ scalar product is $a^i b_i = a\cdot b$

unitarity $\quad U_k{}^i U^k{}_j = U_{ki}\, U^*_{kj} = U_{ki} U^+_{jk} = \delta_{ij} \equiv \delta^i{}_j$

Def. a^i's form a basis for the fundamental (defining) representation of $SU(n)$, denoted $\mathbf{n}$

a_i's form a basis for conjugate representation $\bar{\mathbf{n}}$

$\Rightarrow$ can construct any tensor $a^{i_1 \dots i_p}_{j_1 \dots j_q}$

$$a'^{\,i_1 \dots i_p}_{\;j_1 \dots j_q} = u^{i_1}{}_{k_1} \dots u^{i_p}{}_{k_p} \; u_{j_1}{}^{l_1} \dots u_{j_q}{}^{l_q} \; a^{k_1 \dots k_p}_{l_1 \dots l_q}$$

e.g. δ_i^j is invariant, so is the Levi-Civita symbol $\varepsilon_{i_1 \dots i_n}$.

$\Rightarrow$ in general tensors form bases for reducible representations of $SU(n)$.

$\Rightarrow$ to reduce them to irreducible representations note that permutation operator commutes with all u's: $P_{12}\, a^{ij} = a^{ji}$ $\Rightarrow$

$$\Rightarrow P_{12}\, a'^{ij} = P_{12}\, u^i{}_k\, u^j{}_l\, a^{kl} = u^j{}_k\, u^i{}_l\, a^{kl} =$$

$$= (k \leftrightarrow l) = u^j{}_l\, u^i{}_k\, a^{lk} = u^i{}_k\, u^j{}_l\, P_{12}\, a^{kl}$$

$\Rightarrow$ organize all tensors by eigenstates of P_{12}: they could by <u>symmetric</u> & <u>anti-symmetric</u>.

a^{ij} : $S^{ij} = \frac{1}{2}(a^{ij} + a^{ji})$, $A^{ij} = \frac{1}{2}(a^{ij} - a^{ji})$

$\Rightarrow P_{12} S^{ij} = S^{ij}$; $P_{12} A^{ij} = -A^{ij}$

What is this good for?

Take a product of two representations:

$$a^i b^j = \frac{1}{2}(a^i b^j + a^j b^i) + \frac{1}{2}(a^i b^j - a^j b^i)$$

take $SU(3)$ for example: a^i is $\mathbf{3}$, $a^i b^j$ is $3 \otimes 3$.

$\frac{1}{2}(a^i b^j + a^j b^i)$ has 6 indep. components $\Rightarrow$ ~~transfor~~ makes a basis for representation $\mathbf{6}$.

$\frac{1}{2}(a^i b^j - a^j b^i)$ has 3 indep. components

$$\parallel$$

$$\frac{1}{2}\varepsilon^{ijk} \underbrace{\varepsilon_{klm} a^l b^m}_{c_k} \Rightarrow \text{it is } \bar{3}$$

$\Rightarrow$ we showed that $3 \otimes 3 = 6 \oplus \bar{3}$.

$$a^i b_j = \underbrace{\left(a^i b_j - \frac{1}{3}\delta^i_j a^k b_k\right)}_{\text{traceless } 3\times 3 \text{ matrix}} + \underbrace{\frac{1}{3}\delta^i_j a^k b_k}_{1 \text{ (a singlet)}}$$

$\Rightarrow$ 8 d.o.f. freedom $\Rightarrow$ an 8 (adjoint representation)

$\Rightarrow$ $3 \otimes \bar{3} = 8 \oplus \underline{1}$.

$\Rightarrow$ for a^i_j one can decompose the result into traceless & dimension-1 subspaces.

<u>Another example</u> $3 \otimes 3 \otimes 3 = ?$

$$a^i b^j c^k = a^i \left[\tfrac{1}{2}(b^j c^k + b^k c^j) + \tfrac{1}{2}(b^j c^k - b^k c^j) \right] =$$

= lots of algebra ...

Young Tableaux.

~ a method to avoid tedious symmetrization, etc.

S^{ij} (symmetric tensor) $\rightarrow$ $\boxed{i}\boxed{j}$

$a^i \rightarrow \boxed{i}$ ~ representation N for $SU(N)$

$$a_{i_1} = \underbrace{\varepsilon_{i_1 i_2 \ldots i_N}}_{\text{Levi-Civita symbol}} b^{i_2 \ldots i_N} \Rightarrow b^{i_2 \ldots i_N} \sim \text{anti-symmetric}$$

$A^{ij} = \begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$ (anti-symmetrize vertically)

$$\Rightarrow a_i = \left.\begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline\end{array}\right\} N-1 \text{ boxes}$$, $\bar{N}$ representation for $SU(N)$

For $SU(3)$ $a^i b_j - \frac{1}{3}\delta^i_j a^k b_k$ was adjoint $\Rightarrow$

a Physics insight:

$$3 \otimes \bar{3} = \mathbb{1} \oplus 8$$

$q^i(x)$ ← $i = 1,2,3$ color index

$q^i(x)$ = quark field is in a 3 representation of SU(3) color

$\Rightarrow \bar{q}^i(x)$ = anti-quark is $\bar{3}$

$$q \otimes \bar{q} = \underset{\uparrow}{\mathbb{1}} \oplus 8 \leftarrow \text{octet}$$

octet: e.g. gluon, A_μ^a, $a = 1, \ldots, 8$

$\mathbb{1}$: **meson** (color-singlet $q\bar{q}$ state)

$q \otimes q = 3 \otimes 3 = 6 \oplus \bar{3}$ $\Rightarrow$ can't make a color-singlet hadron out of two quarks $\Rightarrow$ no such particles exist

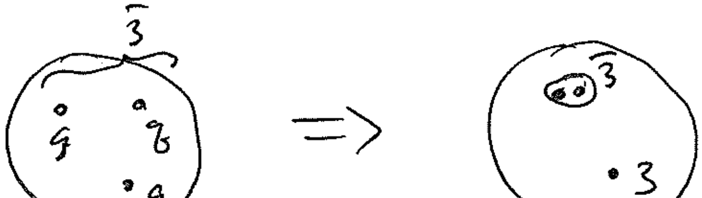
Baryons: $q \otimes q \otimes q = 3 \otimes 3 \otimes 3 =$

$$= (6 \oplus \bar{3}) \otimes 3 = \mathbb{1} \oplus \ldots$$

← pair them up to get a singlet

$\bar{3}$ ↑ two quarks with the anti-quark color $\Rightarrow$ "di-quark"

$\mathbb{1}$: baryons! 🙂



~ color singlet.