Last time | Tensor Method for $SU(N)$

~ label representations by the vectors ("states") they act upon:

$a^i, \quad i=1,\dots,N$ ~ fundamental representation N

$a_i = a^{*i}$ ~ conjugate — " — $\bar{N}$

$a^i b_i$ ~ invariant under $SU(N)$ => singlet $\mathbb{1}$.

Can construct an arbitrary tensor

$$a^{i_1\dots i_p}_{j_1\dots j_q} \rightarrow a'^{i_1\dots i_p}_{j_1\dots j_q} = U^{i_1}_{k_1}\dots U^{i_p}_{k_p}\, U_{j_1}^{l_1}\dots U_{j_q}^{l_q}\, a^{k_1\dots k_p}_{l_1\dots l_q}$$

$U_i^{\ j} = U_{ij}$ ~ $SU(N)$ matrix, $\quad U^i_{\ j} = U^*_{ij}$

Can symmetrize & anti-symmetrize the upper and lower indices separately obtaining irreducible representations.

$SU(3)$ | $a^i b^j = \frac{1}{2}(a^i b^j + a^j b^i) + \frac{1}{2}(a^i b^j - a^j b^i)$

$$\Rightarrow \quad 3 \otimes 3 = 6 \oplus \bar{3}$$

$a^i b_j = \left(a^i b_j - \frac{1}{3}\delta^i_j\, a^k b_k\right) + \frac{1}{3}\delta^i_j\, a^k b_k \Rightarrow 3 \otimes \bar{3} = 8 \oplus \mathbb{1}.$

a^{ij} : $S^{ij} = \frac{1}{2}(a^{ij} + a^{ji})$, $A^{ij} = \frac{1}{2}(a^{ij} - a^{ji})$

$\Rightarrow P_{12} S^{ij} = S^{ij}$; $P_{12} A^{ij} = -A^{ij}$

What is this good for?

Take a product of two representations:

$$a^i b^j = \frac{1}{2}(a^i b^j + a^j b^i) + \frac{1}{2}(a^i b^j - a^j b^i)$$

take $SU(3)$ for example: a^i is $\mathbf{3}$, $a^i b^j$ is $3 \otimes 3$.

$\frac{1}{2}(a^i b^j + a^j b^i)$ has 6 indep. components $\Rightarrow$ ~~transfor~~

makes a basis for representation $\mathbf{6}$.

$\frac{1}{2}(a^i b^j - a^j b^i)$ has 3 indep. components

$$\parallel$$

$$\frac{1}{2} \varepsilon^{ijk} \underbrace{\varepsilon_{klm} a^l b^m}_{c_k} \Rightarrow \text{it is } \bar{3}$$

$\Rightarrow$ we showed that $3 \otimes 3 = 6 \oplus \bar{3}$.

$$a^i b_j = \underbrace{\left(a^i b_j - \frac{1}{3} \delta^i_j a^k b_k\right)}_{\text{traceless } 3\times3 \text{ matrix}} + \underbrace{\frac{1}{3} \delta^i_j a^k b_k}_{1 \text{ (a singlet)}}$$

$\Rightarrow$ 8 d.of.freedom $\Rightarrow$ an 8 (adjoint representation)

$\Rightarrow$ $3 \otimes \bar{3} = 8 \oplus 1$.

$\Rightarrow$ for a^i_j one can decompose the result onto traceless & dimension-1 subspaces.

Another example: $3 \otimes 3 \otimes 3 = ?$

$$a^i b^j c^k = a^i \left[\tfrac{1}{2}(b^j c^k + b^k c^j) + \tfrac{1}{2}(b^j c^k - b^k c^j) \right] =$$

= lots of algebra ...

Young Tableaux.

~ a method to avoid tedious symmetrization, etc.

S^{ij} (symmetric tensor) $\rightarrow$ $\boxed{i}\boxed{j}$

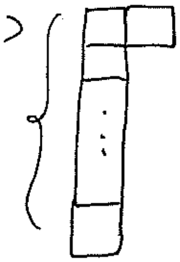
$a^i \rightarrow \boxed{i}$ ~ representation N for $SU(N)$

$$a_{i_1} = \underbrace{\varepsilon_{i_1 i_2 \dots i_N}}_{\text{Levi-Civita symbol}} b^{i_2 \dots i_N} \Rightarrow b^{i_2 \dots i_N} \sim \text{anti-symmetric}$$

$$A^{ij} = \begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array} \quad \text{(anti-symmetrize vertically)}$$

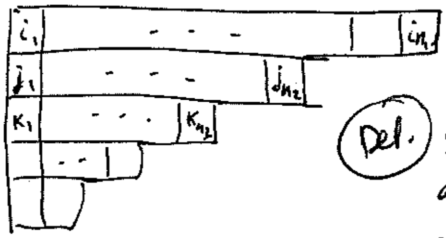
$$\Rightarrow a_i = \left.\begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right\} N-1 \text{ boxes}, \quad \bar{N} \text{ representation for } SU(N)$$

For $SU(3)$ $\quad a^i b_j - \frac{1}{3}\delta^i_j a^k b_k$ was adjoint $\Rightarrow$

$\Rightarrow$ take $a^i b_j \Rightarrow$  ~ adjoint Young tableaux (see this later)

$N-1$ boxes

$\Rightarrow$ general rule: symmetrize in the indices in the rows; then anti-symmetrize in the indices in the columns.



$n_1 \geq n_2 \geq n_3 \ldots$

Def. standard tableaux: index # does not decrease in each row (left to right) and increases in each column.

Example: SU(2): 2 is $a^i \Rightarrow \square$

$\bar{2}$ is $a_i \Rightarrow \square$ too $(N-1=2)$

$\Rightarrow$ 2 and $\bar{2}$ are equivalent.

adjoint representation: $\square\square$ ~ $a^i b_j - \frac{1}{3}\delta^i_j a^k b_k$

$\Rightarrow$ has dimension $3 \Rightarrow$ triplet

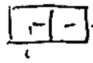
SU(3): 3 is $a^i \Rightarrow \square$, $\bar{3}$ is $a_i \Rightarrow \begin{smallmatrix}\square\\ \square\end{smallmatrix}$.

adjoint representation 8 is $\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$. (octet).

The dimension of a SU(N) representation given by Young tableaux is:

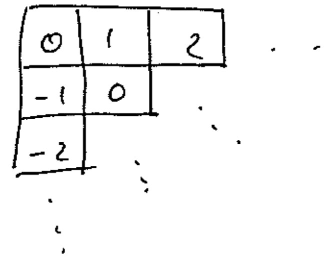
$$\boxed{d = \prod_{\substack{\text{all boxes}\\ i}} (N + D_i)/h_i}$$

See p. 194 in Georgi

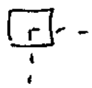
h_i ~ hook length  $h_i = 2$,  $h_i = 1$

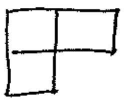
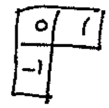
(# boxes crossed by dashed line)  $h_i = 3$

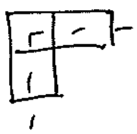
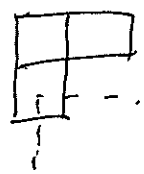
D_i ~ distance to 1st box :

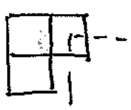


<u>Examples</u> | □ in $SU(N)$

 $h_i = 1$ $\quad \boxed{0} \Rightarrow D_i = 0 \Rightarrow d = N$ ~ true !

$SU(3)$:  $N = 3$ for $SU(3)$ 

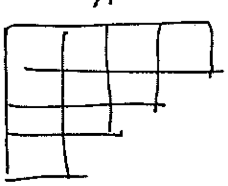
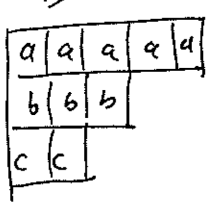
 $h_1 = 3$, $D_1 = 0$  $h_2 = 1$, $D_2 = -1$

 $h_3 = 1$, $D_3 = 1$

$$\Rightarrow d = \prod_{i=1}^{3} \frac{\overset{=3}{N} + D_i}{h_i} = \frac{3}{3} \cdot \frac{2}{1} \cdot \frac{4}{1} = 8$$ ~ works !

<u>Reduction of Product-Representations:</u>

A $\qquad\qquad$ B

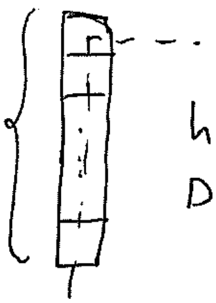
 ⊗ 

(i) take boxes a and put them on the tableaux A building right & down (# boxes in rows should always not increase as you go down & # boxes in columns check:

k boxes $\Rightarrow \binom{N}{k} = \frac{N!}{k!(N-k)!} = \frac{N(N-1)\cdots(N-k+1)}{1\cdot 2\cdot\ldots\cdot k}$

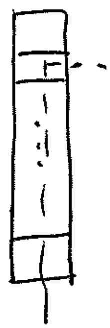
↑ # of ways to choose k diff. box values out of N

& impose "standard" order on indices

k 

$h = K$

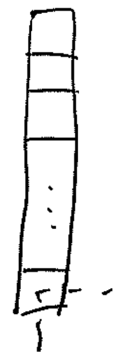
$D = 0$



$h = K-1$

$D = -1$

...



$h = 1$

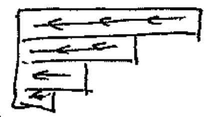
$D = -(K-1)$

$$\Rightarrow \quad d = \frac{N}{K}\cdot\frac{N-1}{K-1}\cdot\ldots\frac{N-K+1}{1} = \binom{N}{K}$$ ~ correct!

does not increase as you go right), no two a's in the same column

(ii) ibid for b's, c's

(iii) to avoid doublecounting, reading from right to left in each row (to make one "phrase" reading all rows top-down) one must have # a's ≥ # b's ≥ # c's ≥ ... to the left of ∀ place in the "phrase":



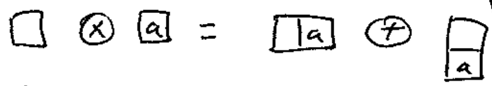
~ read this way

⇓

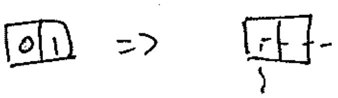
get aababc ...

⇒ should be # a's ≥ # b's ≥ # c's ... to the left from ∀ symbol in the "phrase"

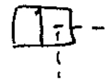
Examples! SU(3):



$3 \otimes 3 = \ ? \oplus \bar{3}$



$h_1 = 2,\ D_1 = 0$

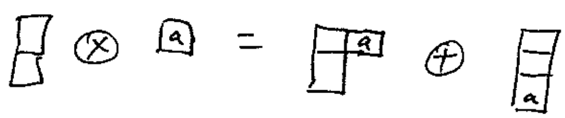


$h_2 = 1,\ D_2 = 1$

$d = \frac{3}{2} \cdot \frac{4}{1} = 6$ ⇒ this is 6! ⇒ $3 \otimes 3 = 6 \oplus \bar{3}$

as we saw before!

What about $3 \otimes \bar{3}$?



$\bar{3} \otimes 3 = 8 \oplus \ ?$

is $\mathbb{1}$ for SU(3) ⇒ a_{ijk} ~ has only one non-trivial component! ~ ε_{ijk}

⇒ $\bar{3} \otimes 3 = 1 \oplus 8$ as before.

= 0 for SU(3) ~ can't have more than 3 boxes in a column

$a_{ijkl} = 0$ if require it to be anti-symmetric.

⇒ get a column of length N for SU(N) ~ a singlet (discard it if it's a part of a larger tableaux)

We wanted to find $3 \otimes 3 \otimes 3$. We know that

$$3 \otimes 3 = 6 \oplus \bar{3} \iff \square \otimes \square = \square\square \oplus \begin{array}{c}\square\\\square\end{array}$$

$$6 \otimes 3 = \square\square \otimes \boxed{a} = \square\square\boxed{a} \oplus \begin{array}{l}\square\square\\\boxed{a}\end{array}$$

$$\qquad\qquad\qquad\qquad\quad ? \qquad\qquad 8$$

$\boxed{0}\boxed{1}\boxed{2}$

$\boxed{1}\,\square\,\square$ $h_1 = 3$, $D_1 = 0$ $\qquad \square\boxed{1}\,\square$ $h_2 = 2$, $D_2 = 1$, $\qquad \square\square\boxed{1}$ $h_3 = 1$, $D_3 = 2$

$$d = \frac{3}{3} \cdot \frac{4}{2} \cdot \frac{5}{1} = 10 \Rightarrow \square\square\square \text{ is } \mathbf{10}.$$

see p. 194 in Georgi

$$6 \otimes 3 = 10 \oplus 8$$

$$3 \otimes \bar{3} = 1 \oplus 8 \qquad \text{(know from before)}$$

$$\Rightarrow \boxed{3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10}$$

check: $3^3 = 27 \qquad 1 + 8 + 8 + 10 = 27$ too!

$\Rightarrow$ to learn more about groups read

"Lie Algebras in Particle Physics"

by H. Georgi

$\Rightarrow$ may want to learn about weights & roots...