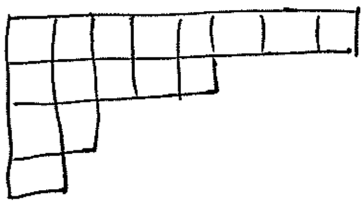
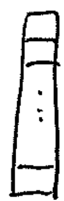
Last time | Young Tableaux



n_1
n_2
n_3
$\vdots$

$n_1 \geq n_2 \geq n_3 \geq \dots$

symmetrize in each row, anti-symmetrice in each column

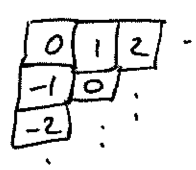
$SU(N)$:  $= N$,  $\}$ $N-1$ boxes $= \bar{N}$

$N-1$ $\{$  $= N^2 - 1$ adjoint representation

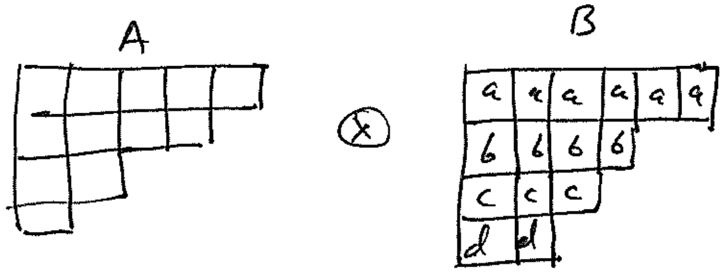
<u>Dimension</u> of a representation given by Young tableax in $SU(N)$ is

$$\boxed{d = \prod_{\substack{\text{all boxes} \\ i}} \frac{N + D_i}{h_i}}$$

h_i = hook length  $h_i = 3$

D_i = distance to 1^{st} box : 

Reduction of Product-Representations:



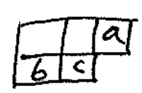
(i) take boxes a and build tableaux A right & down

(ii) ditto for b, c, d, etc.

(iii) read right to left starting from top row, etc. get a single "phrase" aabbc... . In this "phrase" one should have # a's $\geq$ # b's $\geq$ # c's $\geq$ ... to the left of $\forall$ symbol

e.g.  is illegal since

the "phrase" is ba $\Rightarrow$ b is left of a

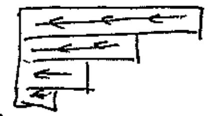
similarly,  has "acb" $\Rightarrow$

also illegal, since # b < # c to the left of b.

does not increase as you go right), no two a's in the same column

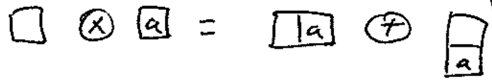
(ii) ibid for b's, c's

(iii) to avoid double counting, reading from right to left in each row one must have # a's $\geq$ # b's $\geq$ # c's $\geq$ ... to the left of $\forall$ place in the "phrase" — to make one "phrase" reading all rows top-down:

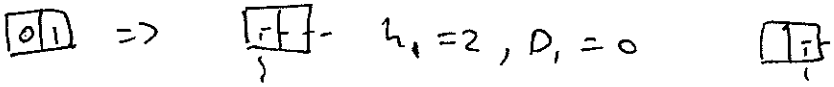


$\Downarrow$ get aababc... $\Rightarrow$ should be # a's $\geq$ # b's $\geq$ # c's ... to the left from $\forall$ symbol in the "phrase"

<u>Examples!</u> $SU(3)$:



$3 \otimes 3 = \ ? \oplus \bar{3}$

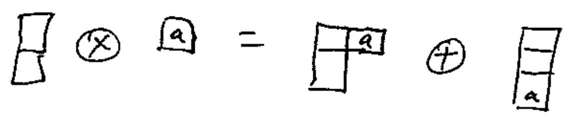


$h_1 = 2,\ D_1 = 0$ $\quad$ $h_2 = 1,\ D_2 = 1$

$d = \frac{3}{2} \cdot \frac{4}{1} = 6 \Rightarrow$ this is 6! $\Rightarrow$ $3 \otimes 3 = 6 \oplus \bar{3}$

as we saw before!

What about $3 \otimes \bar{3}$?



$\bar{3} \otimes 3 = 8 \oplus \ ?$

is $\mathbb{1}$ for $SU(3)$ $\Rightarrow$ a_{ijk} ~ has only one non-trivial component! ~ ε_{ijk}

$\Rightarrow$ $\bar{3} \otimes 3 = 1 \oplus 8$ as before.

$= 0$ for $SU(3)$ ~ can't have more than 3 boxes in a column

$a_{ijkl} = 0$ if require it to be anti-symmetric.

$\Rightarrow$ get a column of length N for $SU(N)$ ~ a singlet (discard it if it's a part of a larger tableaux)

We wanted to find $3 \otimes 3 \otimes 3$. We know that

$$3 \otimes 3 = 6 \oplus \bar{3} \iff \square \otimes \square = \square\square \oplus \begin{array}{c}\square\\ \square\end{array}$$

$$6 \otimes 3 = \square\square \otimes \boxed{a} = \underset{?}{\square\square\boxed{a}} \oplus \underset{8}{\begin{array}{l}\square\square\\ \boxed{a}\end{array}}$$

$\boxed{0}\boxed{1}\boxed{2}$

$\boxed{1}\boxed{-}\boxed{-}$ $h_1 = 3,\ D_1 = 0$ $\square\boxed{1}\boxed{-}$ $h_2 = 2,\ D_2 = 1,$ $\square\square\boxed{1}$ $h_3 = 1,\ D_3 = 2$

$$d = \frac{3}{3} \cdot \frac{4}{2} \cdot \frac{5}{1} = 10 \Rightarrow \square\square\square \text{ is } \mathbf{10}.$$

see p. 194 in Georgi

$$6 \otimes 3 = 10 \oplus 8$$

$$3 \otimes \bar{3} = 1 \oplus 8 \quad \text{(know from before)}$$

$$\Rightarrow \boxed{3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10}$$

check: $3^3 = 27$ $\quad 1 + 8 + 8 + 10 = 27$ too!

⇒ to learn more about groups read

"Lie Algebras in Particle Physics"

by H. Georgi

⇒ may want to learn about weights & roots...

SU(3):

$$3 \otimes \bar{3} = \begin{ytableau} \ \end{ytableau} \otimes \begin{ytableau} a \\ b \end{ytableau} = \begin{ytableau} \ & a & b \end{ytableau} \oplus \begin{ytableau} \ & a \\ b \end{ytableau} \oplus \begin{ytableau} \ \\ a \\ b \end{ytableau} \oplus \begin{ytableau} \ & b \\ a \end{ytableau}$$

consider $\begin{ytableau} \ & a & b \end{ytableau} \leftarrow \Rightarrow$ rule (iii) gives a "phrase"

ba $\Rightarrow$ illegal permutation, since # a < # b

to the left of a $\Rightarrow$ discard this tableaux

$\begin{ytableau} \ & b \\ a \end{ytableau}$ ~ same story, the "phrase" is ba

$\Rightarrow$ illegal $\Rightarrow$ drop

$$\Rightarrow \quad 3 \otimes \bar{3} = \begin{ytableau} \ & a \\ b \end{ytableau} \oplus \begin{ytableau} \ \\ a \\ b \end{ytableau} = 8 \oplus 1 \quad \sim \text{correct!}$$

Quark Symmetries Revisited.

$\Rightarrow$ Isospin symmetry: we had an isospin operator $\vec{I}$ which was like angular momentum operator in 3d isospin fictitious space; as angular momentum it satisfied:

$$[I_a, I_b] = i\,\varepsilon_{abc}\, I_c$$

Compare with SU(2): group elements were $e^{i\vec{\alpha}\cdot\vec{J}}$ with $\vec{J} = \frac{1}{2}\vec{\sigma}$. We had the Lie algebra for generators:

$$[J_i, J_j] = i\,\varepsilon_{ijk}\, J_k .$$

$\Rightarrow$ isospin symmetry is an <u>SU(2) symmetry</u>!

Fundamental representation $\square$ of SU(2) is 2

$\Rightarrow$ a doublet $\Rightarrow$ we saw a lot of isospin doublets

$$\begin{pmatrix} p \\ n \end{pmatrix}, \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix}, \dots$$

$$2 \otimes \bar{2} = \square \otimes \square = \underbrace{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\substack{\shortparallel \\ 1 \text{ (singlet)}}} \oplus \underbrace{\square\square}_{\substack{\shortparallel \\ 3 \text{ (triplet)}}} = 1 \oplus 3$$

$\Rightarrow$ can have isospin 3 $\sim$ triplets: $\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}, \begin{pmatrix} \rho^+ \\ \rho^0 \\ \rho^- \end{pmatrix}, \begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix}, \dots$

⇒ can also have iso-singlets: $\eta, \omega, \phi, \Lambda, \ldots$

- Is this a symmetry of the quark Lagrangian that we wrote? Look at 2 flavors:

$$q(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix} \quad ; \quad m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \quad \Rightarrow$$

$$\Rightarrow \mathcal{L}_{\text{quarks}}^{N_f=2} = \bar{q}\left(i\gamma\cdot\partial - m\right)q$$

First put $m=0 \Rightarrow \mathcal{L}_0 = \bar{q}^f \, i\gamma\cdot\partial \, q^f$

⇒ SU(2) flavor transformation would be

$$e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} \Rightarrow q \to q' = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} q \;, \quad \vec{\alpha} \sim \text{const} \; (x\text{-indep.})$$

$$\Rightarrow \bar{q} \to \bar{q}' = \bar{q} e^{-i\vec{\alpha}\frac{\vec{\sigma}}{2}} \quad \Rightarrow \mathcal{L}_0 \text{ is invariant:}$$

$$\bar{q}\, i\gamma\cdot\partial q \to \bar{q}'\, i\gamma\cdot\partial q' = \bar{q}\, e^{-i\vec{\alpha}\frac{\vec{\sigma}}{2}}\, i\gamma\cdot\partial\, e^{i\vec{\alpha}\frac{\vec{\sigma}}{2}} q =$$

$$= \bar{q}\, i\gamma\cdot\partial q.$$

What about the mass term?

$$\bar{q} m q \to \bar{q}' m q' = \bar{q}\, e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\, m\, e^{i\vec{\alpha}\frac{\vec{\sigma}}{2}} q$$

Write $m = \begin{pmatrix} \frac{m_u+m_d}{2} & 0 \\ 0 & \frac{m_u+m_d}{2} \end{pmatrix} + \begin{pmatrix} \frac{m_u-m_d}{2} & 0 \\ 0 & -\frac{m_u-m_d}{2} \end{pmatrix} \Rightarrow$

$$\Rightarrow \quad m = \frac{m_u + m_d}{2}\,\mathbb{1} + \frac{m_u - m_d}{2}\,\sigma^3$$

$$\Rightarrow \quad \bar{q}'\, m\, q' = \bar{q}\,\frac{m_u+m_d}{2}\,q + \frac{m_u - m_d}{2}\,\bar{q}\,\underbrace{e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\,\sigma^3\, e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}}_{\neq\ \sigma^3}\,q$$

$\Rightarrow$ if $m_u = m_d$ $\Rightarrow$ get exact $SU(2)$ flavor symmetry (global $SU(2)$ symmetry ~ $\vec{\alpha}$ is independent of x^μ!)

as $m_u \neq m_d$ by a little bit $\Rightarrow$ $SU(2)$ flavor is (slightly) broken. ($\Rightarrow$ hadron masses are different)

$\Rightarrow$ in reality the symmetry group is much larger!

~ $SU(2)_R \times SU(2)_L$ ~ more on this later. ($m_u = m_d = 0$)
(for massless quarks)

$\Rightarrow$ Now, put the strange quark back in:

$$q = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix}, \qquad m = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{L}^{N_f=3}_{\text{quarks}} = \bar{q}\,(i\gamma\cdot\partial - m)\,q \qquad \text{again.}$$

$\Rightarrow$ one can check that if $m_u = m_d = m_s$ then $\mathcal{L}$ is invariant under $SU(3)$ flavor transform:

$$q \to q' = e^{i\vec{\alpha}\cdot\vec{t}}\,q\,, \qquad t^a = \tfrac{1}{2}\lambda^a, \quad \lambda^a \sim \text{Gell-Mann matrices}$$

$a = 1, 2, \ldots, 8.$

$\Rightarrow$ as $m_u \neq m_d \neq m_s$ $SU(3)$ is not an exact flavor symmetry.

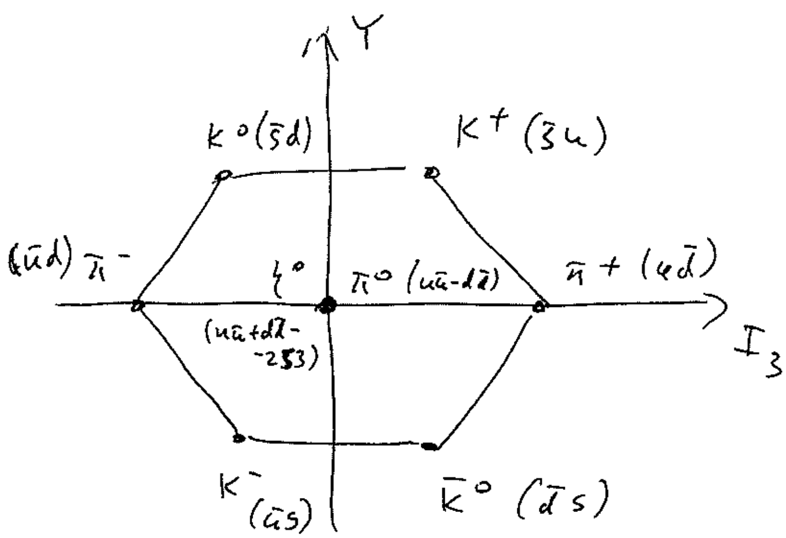
Now, let's look at mesons: $\bar{q}q$ ~ states

$\Rightarrow$ $3 \otimes \bar{3} = 1 \oplus 8$ $\Rightarrow$ there should be a flavor octet and singlet:

pseudoscalar mesons $\pi^+, \pi^-, \pi^0, \eta^0, K^+, K^0, \bar{K}^0, K^-$ form flavor-octet! "The Eightfold Way"!

η' ~ flavor singlet! $\sim (\bar{u}u + \bar{d}d + \bar{s}s)\frac{1}{\sqrt{3}}$.

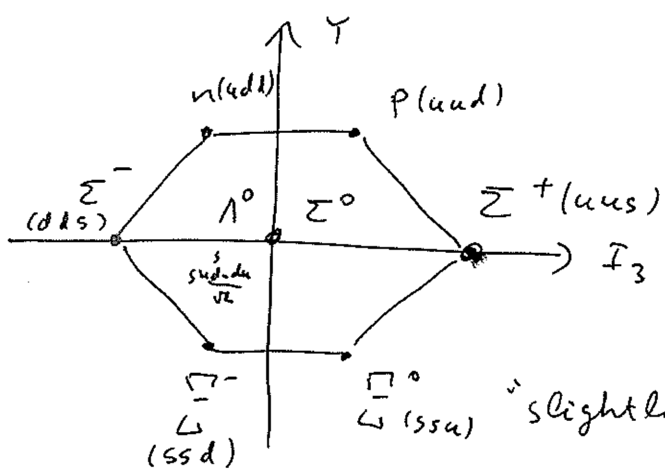
vector mesons ~ the same story!



What about baryons? qqq-states $\Rightarrow$

$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$

$\frac{1}{2}^+$ baryons: $p, n, \Sigma^+, \Sigma^-, \Sigma^0, \Lambda^0, \Xi^0, \Xi^-$ ~ form an octet!



baryon decuplet ~ that's the 10!

$\Rightarrow$ as $SU(3)$ flavor is not exact, all masses are "slightly" different ~ broken symmetry!

$(m_{\Xi^0} = 1315\ \text{MeV},\ m_p = 938\ \text{MeV})$

Gell-Mann – Okubo Mass Formula

⇒ Note that $m_p \neq 2m_u + m_d$ ⇒ most of the mass is due to gluonic interactions ⇒

⇒ write

$m_d \simeq m_u$

$$m_p = m_0 + 2m_u + m_d \simeq m_0 + 3m_u$$

$$m_\Sigma = m_0 + 2m_u + m_s$$

$\Sigma^+ = uus$

$\Xi^0 = uss$

$\Lambda^0 = uds$

$$m_\Xi = m_0 + m_u + 2m_s$$

$m_\Lambda = m_\Sigma$

$$m_\Lambda = m_0 + 2m_u + m_s$$

⇒ $$\frac{m_\Sigma + 3m_\Lambda}{2} = m_p + m_\Xi$$ for $\frac{1}{2}^+$ baryon octet.

$m_p = 938\,MeV$, $m_\Lambda = 1116\,MeV$, $m_\Xi = 1315\,MeV$, $m_\Sigma = 1189\,MeV$

$LHS = 2268.5\,MeV$, $RHS = 2253\,MeV$ ~ close enough!

For $\frac{3}{2}^+$ baryon decuplet get

$\Omega^- = sss$

$$m_\Omega - m_{\Xi^*} = m_{\Xi^*} - m_{\Sigma^*} = m_{\Sigma^*} - m_{\Delta^*}$$

$\Xi^{*-} = ssd$

$\Sigma^{*+} = suu$

$\Delta^{++} = uuu$

~ also works

~ was used to predict the mass of Ω - baryon.

Flavor SU(2) and SU(3) Symmetries.

- Let's go back to 2-flavor QCD:

$$\mathcal{L}^{N_f=2}_{quarks} = \bar{q}\,(i\gamma\cdot\partial - m)\,q \quad , \quad m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

We saw that if $m_u = m_d$ we have SU(2) flavor symmetry in the lagrangian.

$\Rightarrow$ However, masses of hadrons are much larger than current quark masses ($m_p \gg 2m_u + m_d$).

$\Rightarrow$ the flavor symmetry is more due to the fact that quark masses are small!

$\Rightarrow$ put $m_u = m_d = 0$

$\Rightarrow \mathcal{L} = \bar{q}\; i\gamma\cdot\partial\, q$.

Write $q = q_L + q_R = \underbrace{\frac{1-\gamma_5}{2}q}_{q_L} + \underbrace{\frac{\mathbb{1}+\gamma_5}{2}q}_{q_R}$

$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\{\gamma_5, \gamma^\mu\} = 0$, $\gamma_5^{\,+} = \gamma_5$

$\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\gamma_5\gamma_5 = \mathbb{1}$

Projection operators $P_L = \frac{\mathbb{1}-\gamma_5}{2}$, $P_R = \frac{\mathbb{1}+\gamma_5}{2}$

$\Rightarrow P_L^{\,2} = \left(\frac{\mathbb{1}-\gamma_5}{2}\right)^2 = \frac{\mathbb{1} - 2\gamma_5 + \overbrace{\gamma_5^{\,2}}^{=\mathbb{1}}}{4} = P_L$