SU(3):

$$3 \otimes \bar{3} = \begin{ytableau} \; \end{ytableau} \otimes \begin{ytableau} a \\ b \end{ytableau} = \begin{ytableau} \; & a & b \end{ytableau} \oplus \begin{ytableau} \; & a \\ b \end{ytableau} \oplus \begin{ytableau} \; \\ a \\ b \end{ytableau} \oplus \begin{ytableau} \; & b \\ a \end{ytableau}$$

consider $\begin{ytableau} \; & a & b \end{ytableau} \leftarrow \Rightarrow$ rule (iii) gives a "phrase" ba $\Rightarrow$ illegal permutation, since # a < #b to the left of a $\Rightarrow$ discard this tableaux

$\begin{ytableau} \; & b \\ a \end{ytableau}$ ~ same story, the "phrase" is ba $\Rightarrow$ illegal $\Rightarrow$ drop

$$\Rightarrow 3 \otimes \bar{3} = \begin{ytableau} \; & a \\ b \end{ytableau} \oplus \begin{ytableau} \; \\ a \\ b \end{ytableau} = 8 \oplus 1 \quad \text{~ correct!}$$

Rule (iii): Reading from right to left in each row one goes through all the rows top to bottom. In the end one gets a single "phrase" like "aababc..." or "baabcc...", etc., comprised of the letters from all the rows, collected as described above (right-left, top-down). In this phrase, written down left to right now,

$$\# a\text{'s} \geq \# b\text{'s} \geq \# c\text{'s} \ \ldots$$

to the left of any symbol in the "phrase". If this condition is not satisfied, the tableaux is discarded.

Last time | Quark Symmetries Revisited

Isospin symmetry is SU(2).

$$\mathcal{L}^{N_f=2}_{quarks} = \bar{q}\,[\,i\not{\partial} - m\,]\,q$$

$$q(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}, \quad m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

$q(x) \to e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} q(x).$

$\mathcal{L}^{N_f=2}_{quarks}$ has a global SU(2) symmetry

iff $m_u = m_d$. In real life with $m_u \neq m_d$

the symmetry is slightly brocken ($m_p \neq m_n$).

$$\mathcal{L}^{N_f=3}_{quarks} = \bar{q}\,[\,i\not{\partial} - m\,]\,q$$

$$q(x) = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix}, \quad m = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}.$$

SU(3) global : $q(x) \to e^{i\vec{\alpha}\cdot\vec{t}}\, q(x)$

Only exact for $m_u = m_d = m_s$. Since

$m_u \neq m_d \neq m_s \Rightarrow$ SU(3) flavor symmetry

is slightly brocken.

Gell-Mann – Okubo Mass Formula

⇒ Note that $m_p \neq 2m_u + m_d$ ⇒ most of the mass is due to gluonic interactions ⇒

⇒ write

$$m_p = m_0 + 2m_u + m_d \simeq m_0 + 3m_u \quad (m_d \simeq m_u)$$

$$m_\Sigma = m_0 + 2m_u + m_s$$

$\Sigma^+ = uus$

$\Xi^0 = uss$

$\Lambda^0 = uds$

$$m_\Xi = m_0 + m_u + 2m_s$$

$$m_\Lambda = m_0 + 2m_u + m_s$$

$m_\Lambda = m_\Sigma$

⇒ $\dfrac{m_\Sigma + 3m_\Lambda}{2} = m_p + m_\Xi$ for $\frac{1}{2}^+$ baryon octet.

$m_p = 938\,\text{MeV}$, $m_\Lambda = 1116\,\text{MeV}$, $m_\Xi = 1315\,\text{MeV}$, $m_\Sigma = 1189\,\text{MeV}$

LHS = 2268.5 MeV, RHS = 2253 MeV ~ close enough!

For $\frac{3}{2}^+$ baryon decuplet get

$$m_\Omega - m_{\Xi^*} = m_{\Xi^*} - m_{\Sigma^*} = m_{\Sigma^*} - m_{\Delta^*}$$

$\Omega^- = sss$

$\Xi^{*-} = ssd$

$\Sigma^{*+} = suu$

$\Delta^{++} = uuu$

~ also works

~ was used to predict the mass of Ω^--baryon.

Flavor SU(2) and SU(3) Symmetries.

- Let's go back to 2-flavor QCD:

$$\mathcal{L}_{\text{quarks}}^{N_f=2} = \bar{q}\,(i\gamma\cdot\partial - m)\,q \quad , \quad m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

We saw that if $m_u = m_d$ we have SU(2) flavor symmetry in the Lagrangian.

$\Rightarrow$ However, masses of hadrons are much larger than current quark masses ($m_p \gg 2m_u + m_d$).

$\Rightarrow$ the flavor symmetry is more due to the fact that quark masses are small!

$\Rightarrow$ put $m_u = m_d = 0$

$\Rightarrow$ $\mathcal{L} = \bar{q}\, i\gamma\cdot\partial\, q$.

Write $q = q_L + q_R = \underbrace{\frac{1-\gamma_5}{2}\,q}_{q_L} + \underbrace{\frac{\mathbb{1}+\gamma_5}{2}\,q}_{q_R}$

$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\{\gamma_5, \gamma^\mu\} = 0$, $\gamma_5^\dagger = \gamma_5$

$\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\gamma_5\gamma_5 = \mathbb{1}$

Projection operators $P_L = \frac{\mathbb{1}-\gamma_5}{2}$, $P_R = \frac{\mathbb{1}+\gamma_5}{2}$

$$\Rightarrow P_L^2 = \left(\frac{\mathbb{1}-\gamma_5}{2}\right)^2 = \frac{\mathbb{1} - 2\gamma_5 + \underbrace{\gamma_5^2}_{=\mathbb{1}}}{4} = P_L$$

$P_R^2 = P_R$, $P_R P_L = \frac{1\!\!1+\gamma_5}{2}\frac{1\!\!1-\gamma_5}{2} = \frac{1\!\!1-\gamma_5^2}{4} = 0$.

For massless particles they project on different helicity states. $P_L = \frac{1}{2}\begin{pmatrix} 1\!\!1 & -1\!\!1 \\ -1\!\!1 & 1\!\!1 \end{pmatrix}$, $P_R = \begin{pmatrix} 1\!\!1 & 1\!\!1 \\ 1\!\!1 & 1\!\!1 \end{pmatrix}$.

Now, $\bar{q} = q^+\gamma^0 \Rightarrow q^+ = q^+\left(\frac{1\!\!1-\gamma_5}{2}\right) + q^+\left(\frac{1\!\!1+\gamma_5}{2}\right) = q_L^+ + q_R^+$

$$\Rightarrow \bar{q} = \underbrace{\bar{q}\,\frac{1+\gamma_5}{2}}_{\bar{q}_L} + \underbrace{\bar{q}\,\frac{1\!\!1-\gamma_5}{2}}_{\bar{q}_R} \quad \text{as } \{\gamma_5,\gamma_0\} = 0 .$$

$$\Rightarrow \mathcal{L} = \Big[\bar{q}\,\underbrace{\frac{1\!\!1+\gamma_5}{2} + \overbrace{\bar{q}\,\frac{1\!\!1-\gamma_5}{2}\Big]\, i\gamma\cdot\partial\,\Big[\frac{1\!\!1-\gamma_5}{2}}_{\text{survives}} q + \frac{1\!\!1+\gamma_5}{2}}^{\text{survives}} q\Big]$$

$$\Rightarrow \boxed{\mathcal{L} = \bar{q}_L\, i\gamma\cdot\partial\, q_L + \bar{q}_R\, i\gamma\cdot\partial\, q_R}$$

Now, this Lagrangian is <u>separately</u> invariant under $q_L \to e^{i\vec{\alpha}_L\cdot\frac{\vec{\sigma}}{2}} q_L$ and $q_R \to e^{i\vec{\alpha}_R\frac{\vec{\sigma}}{2}} q_R$

$\Rightarrow$ the net symmetry is $\boxed{SU(2)_L \otimes SU(2)_R .}$ Chiral Symmetry

---

$\Rightarrow$ Now add back the mass term with $m_u = m_d$:

$$-m\bar{q}q = -m\Big[\bar{q}\,\frac{1\!\!1+\gamma_5}{2} + \overbrace{\bar{q}\,\frac{1\!\!1-\gamma_5}{2}\Big]\Big[\underbrace{\frac{1\!\!1-\gamma_5}{2}}}\, q + \frac{1\!\!1+\gamma_5}{2} q}\Big]$$

$= -m\left[\bar{q}_L q_R + \bar{q}_R q_L\right] \Rightarrow$ mixing $\Rightarrow$ need $\vec{\alpha}_R = \vec{\alpha}_L$

$\Rightarrow SU(2)_L \otimes SU(2)_R$ is broken down to $SU(2)$.

What are the conserved currents of $SU(2)_R \otimes SU(2)_L$?

Noether theorem: every continuous symmetry gives a conservation law!

Go back to the massless case:

$$\mathcal{L} = \bar{q}_L i\gamma\cdot\partial q_L + \bar{q}_R i\gamma\cdot\partial q_R$$

$$q_L \underset{SU(2)}{\rightarrow} e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} q_L \Rightarrow \text{if } \vec{\alpha} \text{ is small } q_L \rightarrow \left(1 + i\delta\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}\right) q_L$$
$$= q_L + \delta q_L$$

$\Rightarrow \Delta\mathcal{L} = 0$ as it is a symmetry $\Rightarrow$

$$0 = \Delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta q_L}\Delta q_L + \frac{\delta\mathcal{L}}{\delta\bar{q}_L}\Delta\bar{q}_L + \frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\Delta(\partial_\mu q_L) +$$

$$+ \Delta(\partial_\mu\bar{q}_L)\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)} = \underbrace{\left[\frac{\delta\mathcal{L}}{\delta q_L} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\right]}_{=0\ (EOM)}\Delta q_L + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\Delta q_L\right)$$

$$+ \Delta\bar{q}_L\underbrace{\left[\frac{\delta\mathcal{L}}{\delta\bar{q}_L} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)}\right]}_{=0\ (EOM)} + \partial_\mu\left[\Delta\bar{q}_L\cancel{\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)}}\right] \rightarrow 0 \text{ for our } \mathcal{L}$$

$$\Rightarrow 0 = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\Delta q_L\right)$$

$$\Rightarrow 0 = \partial_\mu\left[\bar{q}_L i\gamma^\mu\Delta q_L\right] = \partial_\mu\left[\bar{q}_L i\gamma^\mu i\frac{\delta\vec{\alpha}\cdot\vec{\sigma}}{2} q_L\right]$$

Def $\Rightarrow$ $\partial_\mu j_L^{i\mu} = 0$ where $\boxed{j_L^{i\mu} = \bar{q}_L \gamma^\mu \frac{\sigma^i}{2} q_L}$

left-handed isospin current.

Def Similarly define $\boxed{j_R^{i\mu} = \bar{q}_R \gamma^\mu \frac{\sigma^i}{2} q_R}$ ~ right-handed isospin current.

$\partial_\mu j_R^{i\mu} = 0$

Alternatively can define

Def $j_\mu^i = j_{L\mu}^i + j_{R,\mu}^i = \bar{q} \gamma_\mu \frac{\sigma^i}{2} q$ ~ vector isospin current

$j_{5\mu}^i = j_{R\mu}^i - j_{L\mu}^i = \bar{q} \gamma_\mu \gamma_5 \frac{\sigma^i}{2} q$ ~ axial vector isospin current

<u>Define</u> charges: $\boxed{Q_{L,R}^i(t) = \int d^3x \; j_{L,R\,0}^i(\vec{x}, t)}$

$$\frac{dQ_L^i(t)}{dt} = \int d^3x \; \frac{\partial j_{L0}^i(\vec{x},t)}{\partial t} = \int d^3x \left[\underbrace{\partial_\mu j_L^{i\mu}}_{=0 \text{ (conserved current)}} - \vec{\nabla}\cdot\vec{j}_L^{\,i}\right]$$

$$= -\int d^3x \; \vec{\nabla}\cdot\vec{j}_L^{\,i} = 0$$ ← surface term

$\Rightarrow$ charges are conserved!

$\Rightarrow$ the charges are <u>generators</u> of $SU(2)_L \otimes SU(2)_R$!

One can show that they form the chiral $SU(2)_L \otimes SU(2)_R$ algebra:

$$\boxed{\begin{aligned} [Q_L^i, Q_L^j] &= i\,\varepsilon_{ijk}\, Q_L^k \\ [Q_R^i, Q_R^j] &= i\,\varepsilon_{ijk}\, Q_R^k \\ [Q_L^i, Q_R^j] &= 0 \end{aligned}}$$

← $SU(2)_L$

← $SU(2)_R$

~ commute with each other.

Let's show how Q_L^i generate $SU(2)_L$ transformations. (114)

Let's calculate $[Q_L^i(t), q_L(t,\vec{x})]$:

$$[Q_L^i(t), q_{L\,\alpha}^{\,a}(\vec{x},t)] = \int d^3x' \, [\bar{q}_L \gamma_0 \frac{\sigma^i}{2} q_L(\vec{x}',t),$$

flavor index 1,2

spinor index 1,2,3,4

$$q_{L\,\alpha}^{\,a}(\vec{x},t)] = \int d^3x' \, [\underbrace{\bar{q}_{L\,\beta}^{\,b}(\vec{x}',t)\,(\gamma^0)_{\beta\delta}}_{q^{+\,b}_{L\,\delta}(\vec{x}',t) \;=\; q^{+\,b}_{\delta'}\left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta}} \left(\frac{\sigma^i}{2}\right)_{bc} \underbrace{q_{L\,\delta}^{\,c}(\vec{x}',t)}_{=\left(\frac{1-\gamma_5}{2}\right)_{\delta\delta''} q^c_{\delta''}},$$

$$q_{L\,\alpha}^{\,a}(\vec{x},t)] =$$

$$= \int d^3x' \cdot \left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta} \left(\frac{1-\gamma_5}{2}\right)_{\delta\delta''} \left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \cdot \left(\frac{\sigma^i}{2}\right)_{bc} \cdot$$

$$\cdot [q^{+\,b}_{\delta'}(\vec{x}',t)\, q^c_{\delta''}(\vec{x}',t),\; q^a_{\alpha'}(\vec{x},t)]$$

$\Rightarrow$ use the anti-commutation relations

$$\{q^a_\alpha(\vec{x},t),\, q^{+\,b}_\beta(\vec{x}',t)\} = \delta^{ab}\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}').$$

$$[Q_L^i(t), q_{L\,\alpha}^{\,a}(\vec{x},t)] = \left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta''} \left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \left(\frac{\sigma^i}{2}\right)_{bc} \cdot \int d^3x' \cdot$$

$$\cdot(-)\{q^a_{\alpha'}(\vec{x},t),\, q^{+\,b}_{\delta'}(\vec{x}',t)\}\, q^c_{\delta''}(\vec{x}',t) = -\left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta''} \cdot$$

$$\cdot\left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \left(\frac{\sigma^i}{2}\right)_{bc} \delta^{ab}\,\delta_{\alpha'\delta'}\, q^c_{\delta''}(\vec{x},t) = -\left(\frac{\sigma^i}{2}\right)_{ac} \cdot$$

$$\cdot\left(\frac{1-\gamma_5}{2}\right)_{\alpha\delta''} q^c_{\delta''}(\vec{x},t) = -\left(\frac{\sigma^i}{2}\right)_{ac} q^c_{L\alpha}(\vec{x},t)$$

$N_f = 2$

$$\boxed{SU(2)_L \otimes SU(2)_R} \qquad m_u = m_d = 0$$

$$\downarrow$$

$$\boxed{SU(2)} \qquad m_u = m_d \neq 0$$

$$\downarrow$$

$$\text{Nothing} \qquad m_u \neq m_d \neq 0$$

Explicitly brocken symmetries in

$$\mathcal{L}^{N_f=2}_{\text{quarks}} = \bar{q}\left[i \not{\partial} - m \right] q .$$