Spontaneous (Chiral) Symmetry Breaking.

General Discussion: Spontaneous Symmetry Breaking.

Def. Spontaneous Symmetry Breaking (SSB):
a symmetry which is manifest in Lagrangian (and Hamiltonian), but is not respected by the ground state of the system.

Example: Ising model in 2 d: $H = -\sum_{\substack{\text{nearest}\\ \text{neighbours}\\ i,j}} \frac{J}{4} S_i S_j$

$J > 0 \Rightarrow$ spins tend to align

$S_i = \pm 1 \sim$ projection of spins on y-axis

$\Rightarrow$ the system is up-down symmetric:

H is invariant under $S_i \to -S_i$.

$\Rightarrow$ However, the system spontaneously choses a ground state, which is either all spins up or all spins down:

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \end{array} \qquad \text{or} \qquad \begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

In this ground state one has $\langle S_i \rangle \neq 0$ ~ non-zero magnetization $\Rightarrow$ $S_i \to -S_i$ invariance is lost.

(Note that $S_i \to -S_i$ is still a symmetry of H!)

<u>Landau - Ginzburg theory of ferromagnetism:</u>

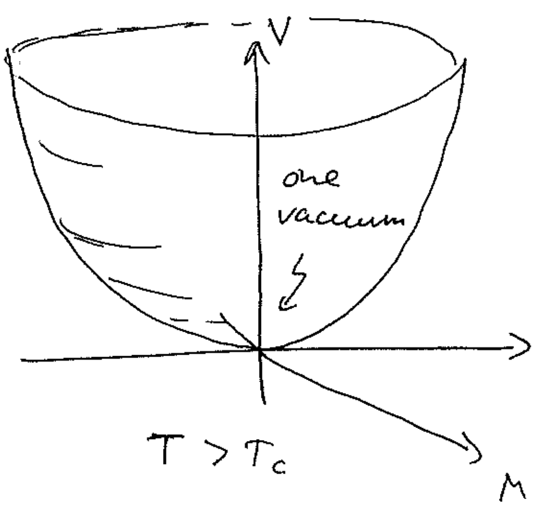
$$H = \int d^3x \left[\left(\nabla_i \cdot M_j\right)^2 + \underbrace{\mu^2 (T - T_c)\, \vec{M}^2 + \lambda \left(\vec{M}^2\right)^2}_{V(\vec{M}) \text{ ~ the potential.}} \right]$$

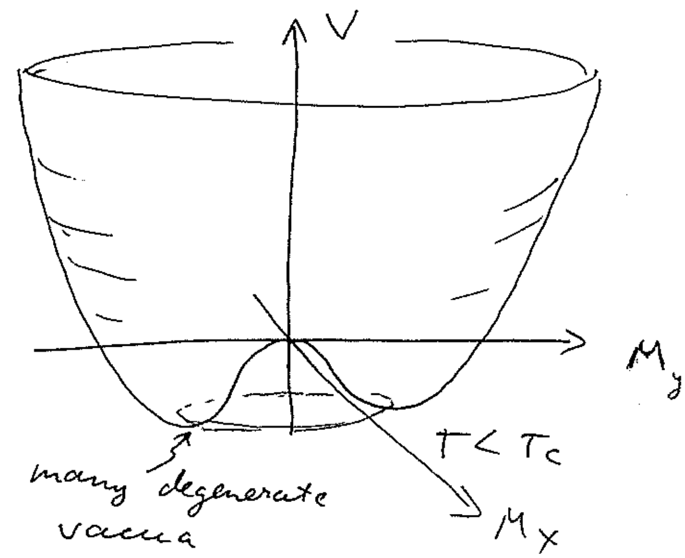
$\vec{M} = (M_1; M_2, M_3)$ is magnetization of the medium

$\lambda > 0$ ~ constant, $\mu^2 > 0$ ~ constant

T ~ temperature, T_c ~ critical (Curie) temperature.

Let's plot the potential $V(\vec{M})$: (assume 2d system)





"Mexican hat" potential

$\Rightarrow$ the Hamiltonian is symmetric under spatial rotations: $M^i \to M'^i = R^{ij} M^j$

$x'_i = R_{ij} x_j \Rightarrow |\vec{x}'|^2 = |\vec{x}|^2 \Rightarrow R_{ij} x_j R_{ik} x_k = x_i x_i$

$\Rightarrow R_{ij} R_{ik} = \delta_{jk} \Rightarrow R \cdot R^T = R^T R = \mathbb{1} \Rightarrow$ forget reflections $\Rightarrow$ require $\det R = +1 \Rightarrow SO(3)$

~ a group of special ($\det = +1$) real orthogonal ($RR^T = R^T R = \mathbb{1}$) 3×3 matrices.

$\Rightarrow$ for $T < T_c$ the ground state is at the minima

$\Rightarrow \quad \mu^2 (T - T_c)\, 2 |\vec{M}| + 4\lambda |\vec{M}|^3 = 0$

$$\Rightarrow \boxed{|\vec{M}_{vac}| = \sqrt{\frac{\mu^2 (T_c - T)}{2\lambda}}}$$

$\Rightarrow$ however, direction of $\vec{M}$ is chosen spontaneously!

Say, $M_{vac} = \sqrt{\dfrac{\mu^2(T_c - T)}{2\lambda}}\ \hat{x} = |0\rangle$

define generators of $SO(3)$: $L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$, $L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$,

$L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow e^{-i \vec{\alpha} \cdot \vec{L}}$ is a rotation by angle $|\vec{\alpha}|$ around $\vec{\alpha}$-direction.

$\Rightarrow H$ is invariant under $\vec{M} \to \vec{M}' = e^{-i\vec{\alpha}\cdot\vec{L}} \vec{M}$.

$\Rightarrow$ ground state is not rotationally symmetric:

$R|0\rangle \neq |0\rangle \Rightarrow$ if $R = e^{i\vec{\alpha}\cdot\vec{Q}}$, $\vec{Q}$ ~ conserved charges of symmetry $\Rightarrow Q^i |0\rangle \neq 0$ $\left(\begin{array}{c}\text{equivalently} \\ \langle 0|\vec{M}|0\rangle \neq 0\end{array}\right)$

General Discussion

Imagine a system with Hamiltonian H and conserved symmetry charges Q^i : $[H, Q^i] = 0$.

Act on vacuum: $H|0\rangle = 0$ (can choose vacuum to be 0-energy state)

$$H Q^i |0\rangle = \underbrace{[H, Q^i]}_{=0} |0\rangle + Q^i \underbrace{H|0\rangle}_{=0} = 0$$

$\Rightarrow H Q^i |0\rangle = 0$ $\Rightarrow$ either

(i) $Q^i|0\rangle = 0$ ~ no broken symmetries, vacuum is invariant under Q^i: $e^{i\vec{\varepsilon}\cdot\vec{Q}}|0\rangle = |0\rangle$.

(ii) $Q^i|0\rangle \neq 0$ $\Rightarrow$ vacuum is degenerate, more than one state such that $H|\psi_0\rangle = 0$.
(e.g. rotating ground state in L-G model would give other possible ground states)
$\Rightarrow$ if the system spontaneously chooses one of these $|\psi_0\rangle$ states for its ground state $\Rightarrow$ spontaneous symmetry breaking.