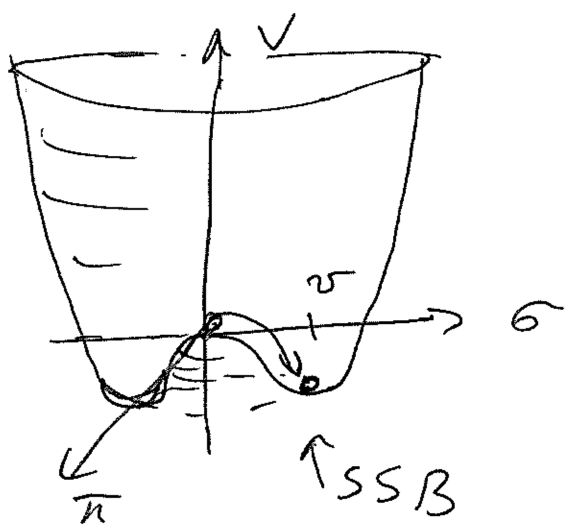
Last time | The Nambu - Goldstone Theorem

(cont'd)

Th'm Spontaneous breakdown of a continuous symmetry implies existence of massless spin-0 particles. (Nambu-Goldstone bosons)

Example 2 | Abelian σ-model:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \sigma\, \partial^\mu \sigma + \frac{1}{2}\partial_\mu \pi\, \partial^\mu \pi + \underbrace{\frac{\mu^2}{2}(\sigma^2+\pi^2) - \frac{\lambda}{4}(\sigma^2+\pi^2)^2}_{-V}$$



$v = \mu\sqrt{\frac{1}{\lambda}} = \langle 0|\sigma|0\rangle$

$\langle 0|\pi|0\rangle = 0$

write $\sigma(x) = v + \sigma'(x)$

$\Rightarrow m_{\sigma'} = \mu\sqrt{2}$

$m_\pi = 0 \sim$ N-G boson

Example 2: Abelian σ-Model:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi + \underbrace{\frac{\mu^2}{2}(\sigma^2+\pi^2) - \frac{\lambda}{4}(\sigma^2+\pi^2)^2}_{-V}$$

with $\mu^2 > 0$, $\lambda > 0$ (constants).

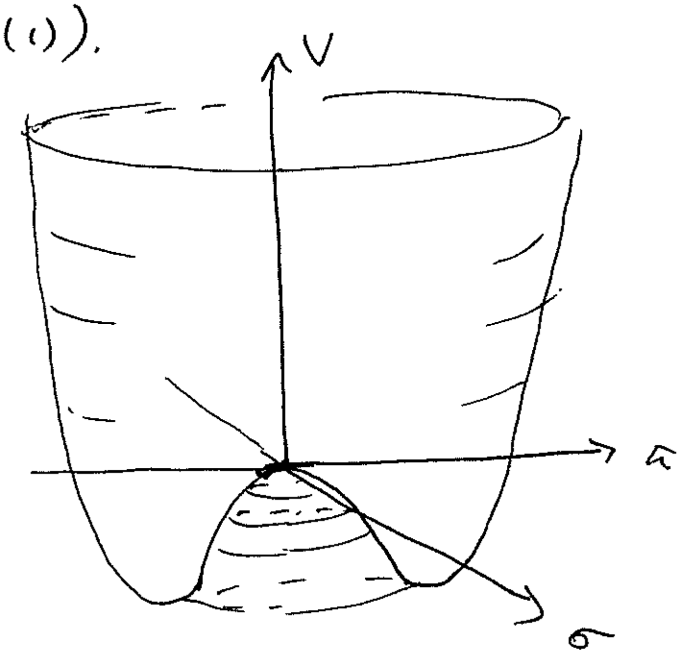
σ, π ~ real fields

$\mathcal{L}$ is invariant under rotations:

$$\begin{pmatrix}\sigma\\ \pi\end{pmatrix} \to \begin{pmatrix}\sigma'\\ \pi'\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\sigma\\ \pi\end{pmatrix}, \quad \alpha \sim \text{real \#}$$

$\Rightarrow$ $O(2)$ symmetry ($= U(1)$).

$\Rightarrow$ get "Mexican hat" potential again:



$\Rightarrow$ the minimum is at

$\sigma^2 + \pi^2 = v^2$

$$\Rightarrow \left(\frac{\mu^2}{2}\cdot v^2 - \frac{\lambda}{4}v^4\right)'_v = 0$$

$\mu^2\cdot v - \lambda v^3 = 0$

$$\Rightarrow \boxed{v = \mu\sqrt{\frac{1}{\lambda}}}$$

$\Rightarrow$ Direction in (σ, π) space is random $\Rightarrow$

$\Rightarrow$ pick the vacuum to be at $\langle 0|\sigma|0\rangle = v$, $\langle 0|\pi|0\rangle = 0$.

Expand $\mathcal{L}$ near the vacuum: $\sigma = v + \sigma'$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu \sigma' \partial^\mu \sigma' + \frac{1}{2}\partial_\mu \bar{\pi}\partial^\mu \bar{\pi} + \frac{\mu^2}{2}\left[(v+\sigma')^2 + \bar{\pi}^2\right]$$

$$- \frac{\lambda}{4}\left[(v+\sigma')^2+\bar{\pi}^2\right]^2 = \frac{1}{2}\partial_\mu\sigma'\partial_\mu\sigma' + \frac{1}{2}\partial_\mu\bar{\pi}\partial^\mu\bar{\pi} + \underset{\text{drop}}{\text{const}}$$

$$+ \sigma'\cancel{\left[\mu^2 v - \frac{\lambda}{4}\cdot 4v^3\right]}^{\to 0} + \sigma'^2\left[\cancel{\frac{\mu^2}{2}} - \frac{\lambda}{4}\cdot(\cancel{2v^2} + 4v^2)\right] +$$

$$+ \bar{\pi}^2\cancel{\left[\frac{\mu^2}{2} - \frac{\lambda}{4}\cdot 2v^2\right]}^{\to 0} - \frac{\lambda}{4}\left[4\sigma' v(\sigma'^2+\bar{\pi}^2) + (\sigma'^2+\bar{\pi}^2)^2\right]$$

$$= \frac{1}{2}\partial_\mu\sigma'\partial_\mu\sigma' + \frac{1}{2}\partial_\mu\bar{\pi}\partial^\mu\bar{\pi} - \mu^2\sigma'^2 - \lambda v\,\sigma'(\sigma'^2+\bar{\pi}^2)$$

$$- \frac{\lambda}{4}(\sigma'^2+\bar{\pi}^2)^2.$$

$\Rightarrow$ now $\bar{\pi}$'s have no $\bar{\pi}^2$ term $\Rightarrow$ $\bar{\pi}$ field is <u>massless</u> in agreement with Goldsone th'm!

$m_{\bar{\pi}} = 0, \quad m_{\sigma'} = \mu\sqrt{2}.$

<u>Non-Abelian σ-Model</u>

Let's illustrate how the chiral $SU(3)_L \otimes SU(3)_R$ symmetry is broken in QCD. As an example consider breaking of $SU(2)_L \otimes SU(2)_R$ symmetry.

Start with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec{\pi}\partial^\mu\vec{\pi}\right) + \frac{\mu^2}{2}(\sigma^2+\vec{\pi}^2) - \frac{\lambda}{4}\left[\sigma^2+\vec{\pi}^2\right]^2$$

$\mu^2, \lambda > 0$, $\quad \overset{\text{isoscalar}}{\sigma}, \underbrace{\vec{\pi} = (\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3)}_{\text{iso triplet, pions}}$ ~ real fields

Define a $\overset{2\times 2}{\vee}$ matrix field $\Sigma = \sigma \mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}$ (128)

τ^1, τ^2, τ^3 ~ Pauli matrices (we use τ to not confuse them with σ)

$$\Rightarrow \operatorname{tr}\left[\Sigma \quad \Sigma^{+}\right] = \operatorname{tr}\left[\sigma^2 \mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}(-i)\,\vec{\tau}\cdot\vec{\pi}\right]$$

$$= 2\sigma^2 + 2\vec{\pi}^{\,2} \qquad \text{as } \operatorname{tr} \tau^i\tau^j = 2\delta^{ij}$$

$$\Rightarrow \operatorname{tr}\left[\partial_\mu \Sigma \ \partial^\mu \Sigma^{+}\right] = 2\left[\partial_\mu \sigma\, \partial^\mu \sigma + \partial_\mu \vec{\pi}\ \partial^\mu \vec{\pi}\right]$$

$$\Rightarrow \mathcal{L}_\Sigma = \frac{1}{4}\left[\operatorname{tr}\ \partial_\mu \Sigma\ \partial^\mu \Sigma^{+}\right] + \frac{\mu^2}{4}\operatorname{tr}\left[\Sigma\Sigma^{+}\right] - \frac{\lambda}{16}\left(\operatorname{tr}\left[\Sigma\Sigma^{+}\right]\right)^2$$

Now add "quarks": (originaly they were protons and neutrons)! $q = \begin{pmatrix} u \\ d \end{pmatrix}$ or $\begin{pmatrix} p \\ n \end{pmatrix} = q^N$

$$\mathcal{L} = \bar{q}^N i\gamma\cdot\partial q^N - g\,\bar{q}^N\left[\sigma\mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right] q^N + \mathcal{L}_\Sigma$$

Such that

$$\mathcal{L} = \bar{q}^N i\gamma\cdot\partial q^N - g\,\bar{q}^N\left[\sigma\mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right] q^N + \frac{1}{2}\left(\partial_\mu\sigma\,\partial^\mu\sigma + \partial_\mu\vec{\pi}\ \partial^\mu\vec{\pi}\right) + \frac{\mu}{2}\left(\sigma^2 + \vec{\pi}^{\,2}\right) - \frac{\lambda}{4}\left(\sigma^2 + \vec{\pi}^{\,2}\right)^2$$

full Lagrangian for $SU(2)_L \otimes SU(2)_R$ σ-model.

(Gell-Mann & Levi, 1960)

As usual write $q^N = q_L^N + q_R^N \Rightarrow$

$$\bar{q}^N i\gamma\cdot\partial q^N = \bar{q}_L^N i\gamma\cdot\partial q_L^N + \bar{q}_R^N i\gamma\cdot\partial q_R^N$$

$$\bar{q}^N\left[\sigma \mathbb{1} + i\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right] q^N = \Big(\underbrace{\bar{q}^N \frac{\mathbb{1}+\gamma_5}{2}}_{\bar{q}_L} + \underbrace{\bar{q}^N \frac{\mathbb{1}-\gamma_5}{2}}_{\bar{q}_R}\Big) \cdot$$

$$\left[\sigma \mathbb{1} + i\,\vec{\tau}\cdot\vec{\pi}\,\gamma_5\right] \Big(\underbrace{\frac{\mathbb{1}-\gamma_5}{2} q^N}_{q_L} + \underbrace{\frac{\mathbb{1}+\gamma_5}{2} q^N}_{q_R}\Big) = \Big| \text{ as } (\gamma_5)^2 = 1$$

$$= \sigma\left[\bar{q}_L^N q_R^N + \bar{q}_R^N q_L^N\right] + i\left[-\bar{q}_R^N \vec{\tau}\cdot\vec{\pi}\, q_L^N + \bar{q}_L^N \vec{\tau}\cdot\vec{\pi}\, q_R^N\right]$$

$$= \bar{q}_L^N \Sigma q_R^N + \bar{q}_R^N \Sigma^+ q_L^N$$

$$\Rightarrow \boxed{\begin{aligned}\mathcal{L} &= \bar{q}_L^N i\gamma\cdot\partial q_L^N + \bar{q}_R^N i\gamma\cdot\partial q_R^N + \frac{1}{4}\mathrm{tr}\left[\partial_\mu \Sigma\, \partial^\mu \Sigma^+\right] \\ &+ \frac{\mu^2}{4}\mathrm{tr}\left[\Sigma\Sigma^+\right] - \frac{\lambda}{16}\left(\mathrm{tr}\left[\Sigma\Sigma^+\right]\right)^2 - \\ &- g\left[\bar{q}_L^N \Sigma q_R^N + \bar{q}_R^N \Sigma^+ q_L^N\right]\end{aligned}}$$

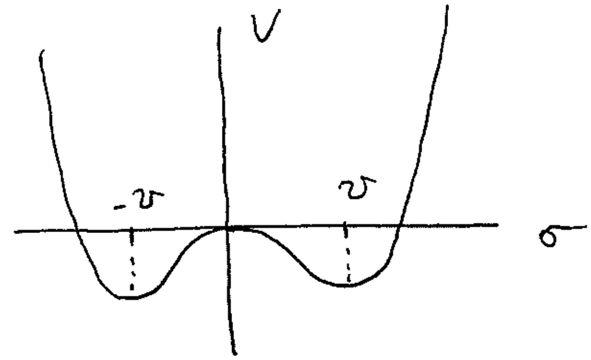
(effective low-energy Lagrangian not QCD, but has the right symmetries)

$\Rightarrow$ this Lagrangian is symmetric under

$$\left[\begin{aligned} q_L^N &\to q_L^{N\prime} = e^{i\vec{\alpha}_L\cdot\frac{\vec{\tau}}{2}} q_L^N \equiv U_L q_L^N \\ q_R^N &\to q_R^{N\prime} = e^{i\vec{\alpha}_R\cdot\frac{\vec{\tau}}{2}} q_R^N \equiv U_R q_R^N \\ \Sigma &\to \Sigma' = U_L \Sigma U_R^+ \end{aligned}\right.$$

$\Rightarrow$ it has $SU(2)_L \otimes SU(2)_R$ symmetry!

For $\mu^2 > 0$ the $SU(2)_L \otimes SU(2)_R$ symmetry is spontaneously broken:



$$\left(\frac{\mu^2}{2}\sigma^2 - \frac{\lambda}{4}\sigma^4\right)' = 0$$

$$\Rightarrow \quad v = \frac{\mu}{\sqrt{\lambda}}$$

$\Rightarrow$ pick $\langle \Psi_0 | \sigma | \Psi_0 \rangle = v$, $\langle \Psi_0 | \vec{\pi} | \Psi_0 \rangle = 0$ as the vacuum.

Write $\sigma = v + \sigma'$ $\Rightarrow$ $\mathcal{L} = \bar{q}^N i\gamma\cdot\partial q^N - g\,\bar{q}^N[v + \sigma' + i\vec{\tau}\cdot\vec{\pi}\,\gamma_5]\,q^N + \frac{1}{2}\left[\partial_\mu \sigma' \partial^\mu \sigma' + \partial_\mu \vec{\pi}\,\partial^\mu \vec{\pi}\right] - \mu^2 \sigma'^2 - \lambda v \sigma'(\sigma'^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma'^2 + \vec{\pi}^2)^2$.

$\Rightarrow$ σ' has mass $\sqrt{2}\mu$.

$\vec{\pi}$ have mass 0. ~ Goldstone bosons (pions)

q^N (proton, neutron) have mass gv. ~ can be large!

Identify $\vec{\pi} \Leftrightarrow \bar{q}\gamma_5\vec{\tau}q$ (q now are real quarks)

$\sigma \Leftrightarrow \bar{q}q$

q^N ~ proton, neutron ~ nucleons

$\Rightarrow$ $SU(2)_L \otimes SU(2)_R$ is spontaneously broken down to $SU(2)$

$\Rightarrow$ pions (π^+, π^-, π^0) are Goldstone bosons of chiral SSB, $m_\pi = 0$ (SU(2) has 3 generators $\Rightarrow$ 3 pions!)

$\Rightarrow$ protons, neutrons get a mass $m_N = g v$ which is large.

$\Rightarrow$ if $SU(2)_L \otimes SU(2)_R$ was exact would have $m_\pi = 0$

but as $m_u \neq m_d \neq 0$ $SU(2)_L \otimes SU(2)_R$ is explicitly broken too $\Rightarrow$ get massive pions!

$\Rightarrow$ for $N_f = 3$ have $SU(3)_L \otimes SU(3)_R$ broken down spontaneously to SU(3) flavor.

$\Rightarrow$ SU(3) has 8 symmetry charges

φ^a, $a = 1, \ldots, 8$

$\Rightarrow$ have 8 Goldstone bosons:

$\pi^+, \pi^-, \pi^0, K^+, K^0, \bar{K}^0, K^-, \eta^0$

$\Rightarrow$ $SU(3)_L \otimes SU(3)_R$ is also badly broken explicitly as $m_s \neq m_u \neq m_d \neq 0$ $\Rightarrow$ K's & η are also massive!

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what is v (VEV) in QCD? Remember $\sigma = \bar{q} q$ $\Rightarrow$

$v = \langle 0 | \bar{q} q | 0 \rangle \simeq -(230\ \text{MeV})^3$ quark condensate or chiral condensate.

$m_\pi^2 \sim (m_u + m_d) \langle 0 | \bar{q} q | 0 \rangle$.