Last time | Non-Abelian σ-model

$$\boxed{\mathcal{L} = \bar{q}^N i\partial\!\!\!/ q^N - g\,\bar{q}^N\left[\mathbb{1}\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5\right]q^N + \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}\right) + \frac{\mu^2}{2}\left(\sigma^2+\vec{\pi}^2\right) - \frac{\lambda}{4}\left(\sigma^2+\vec{\pi}^2\right)^2}$$

$$q^N = \begin{pmatrix} p \\ n \end{pmatrix}$$

Define $\Sigma = \mathbb{1}\sigma + i\vec{\tau}\cdot\vec{\pi}$

$$\Rightarrow \boxed{\mathcal{L} = \bar{q}_L^N i\partial\!\!\!/ q_L^N + \bar{q}_R^N i\partial\!\!\!/ q_R^N + \frac{1}{4}\mathrm{tr}\left[\partial_\mu\Sigma\,\partial^\mu\Sigma^\dagger\right] + \frac{\mu^2}{4}\mathrm{tr}\left[\Sigma\Sigma^\dagger\right] - \frac{\lambda}{16}\left(\mathrm{tr}\left[\Sigma\Sigma^\dagger\right]\right)^2 - g\left[\bar{q}_L^N\Sigma q_R^N + \bar{q}_R^N\Sigma^\dagger q_L^N\right]}$$

$SU(2)_L \times SU(2)_R$ symmetry:

$$\left[\begin{array}{l} q_L^N \to U_L q_L^N \\ q_R^N \to U_R q_R^N \\ \Sigma \to U_L \Sigma U_R^\dagger \end{array}\right.$$

The symmetry is spontaneously broken, giving masses to nucleons, but no mass to pions ~ Goldstone bosons.

The Electroweak Theory

the theory of <u>leptons</u> (spin - 1/2): $e \quad \mu \quad \tau$

$\nu_e \quad \nu_\mu \quad \nu_\tau$

<u>gauge bosons</u> (spin - 1): γ, $\underbrace{W^+, W^-, Z.}_{}$ → massive

& <u>Higgs boson</u>: (spin - 0): φ or H

<u>Local Abelian Gauge Symmetry: a Review</u>

Start with quantum electrodynamics (QED).

Take electrons only: $\mathcal{L} = \bar{\psi}\,[\,i\gamma\cdot\partial - m\,]\,\psi$

$\psi \sim$ Dirac field for electrons.

$\mathcal{L}$ is invariant under $\psi \to \psi' = e^{i\alpha}\psi$

$\alpha \sim$ real number (a constant).

$\Rightarrow$ this is a <u>global</u> $U(1)$ symmetry!

<u>Global symmetry</u>: indep. of x^μ.

$\Rightarrow$ Say we want to have $\alpha(x)$: $\psi \to \psi' = e^{i\alpha(x)}\psi(x)$.

Want to have $\mathcal{L}$ invariant under this <u>local U(1) symmetry</u> (<u>local</u> = $\alpha = \alpha(x)$).

$$\bar{\psi}\left[i\gamma\cdot\partial - m\right]\psi \to \bar{\psi}\, e^{-i\alpha(x)}\left[i\gamma\cdot\partial - m\right]e^{i\alpha(x)}\psi$$

$$= \bar{\psi}\left[i\gamma\cdot\partial + i\gamma\cdot\partial(i\alpha) - m\right]\psi = \bar{\psi}\left[i\gamma\cdot\partial - m\right]\psi$$

$$- \bar{\psi}\,\gamma^{\mu}(\partial_\mu\alpha)\psi$$

$\Rightarrow$ $\mathcal{L}$ is not invariant under local $\overset{U(1)}{\vee}$ symmetry!

$\Rightarrow$ Fix it by introducing local gauge field

$A_\mu(x)$ (gauge the Lagrangian): $\left(\begin{matrix} g=-e \\ \text{electron} \\ \text{charge}\end{matrix}\right)$

$$\mathcal{L} = \bar{\psi}\left[i\gamma\cdot\partial - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g\bar{\psi}\gamma^\mu A_\mu\psi$$

$\Rightarrow$ require that: $\boxed{\begin{matrix}\psi \to e^{i\alpha(x)}\psi \\ A_\mu \to A_\mu + \frac{1}{g}\partial_\mu\alpha(x)\end{matrix}}$

$$\Rightarrow \mathcal{L} \to \mathcal{L}' = \bar{\psi}\left[i\gamma\cdot\partial - m\right]\psi - \cancel{\bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$+ g\bar{\psi}\gamma^\mu A_\mu\psi + \cancel{\bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi} = \mathcal{L}$$

$\Rightarrow$ now it is invariant!

$\Rightarrow$ Def. Covariant derivative $D_\mu \equiv \partial_\mu - igA_\mu$

$$\Rightarrow \mathcal{L}_{QED} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

as usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F_{\mu\nu} = \left[D_\mu D_\nu - D_\nu D_\mu\right]\frac{i}{g} = \frac{i}{g}\left[D_\mu, D_\nu\right]$

SU(2) Gauge Theory

Now ~~imagine a theory~~ with a non-abelian symmetry, like SU(2): $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $\psi_1, \psi_2 \sim$ spinors

ψ_1 & ψ_2 are different by some quantum #
(e.g. color, weak isospin)

$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m\right]\psi$ with $m = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ is invariant under $\psi \to \psi' = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\psi$

$\vec{\sigma}$ are Pauli matrices in $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ space.

$\Rightarrow$ global SU(2) symmetry.

$\Rightarrow$ let's make it local (gauge it): $\vec{\alpha} = \vec{\alpha}(x)$

$$\Rightarrow \psi \to \psi' = e^{i\vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2}}\psi(x) \equiv S(x)\,\psi(x)$$

with $S^+ S = S S^+ = \mathbb{1}$.

$$\Rightarrow \mathcal{L} \to \mathcal{L}' = \bar{\psi} S^+\left[i\gamma^\mu \partial_\mu - m\right] S \psi = \bar{\psi}\left[i\gamma\cdot\partial - m\right]\psi$$

$$+ \bar{\psi}\, i\gamma^\mu \left(S^+ \partial_\mu S\right)\psi \quad \Rightarrow \text{not invariant}$$

$\Rightarrow$ add a gauge field A_μ^a, $a = 1, 2, 3$:

$$\mathcal{L} = \bar{\psi}\left[i\gamma\cdot\partial - m\right]\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + g\bar{\psi}\gamma^\mu A_\mu^a \frac{\sigma^a}{2}\psi$$

$$\mathcal{L} \to \mathcal{L}' = \bar{\psi}\,[i\gamma^\mu \partial_\mu - m]\,\psi + \bar{\psi}\, i\gamma^\mu (S^+ \partial_\mu S)\,\psi +$$

$$+\, g\,\bar{\psi}\,\gamma^\mu S^+ A'_\mu S\,\psi - \frac{1}{4} F'^{a}_{\mu\nu} F'^{a\mu\nu}.$$

where $A_\mu = A_\mu^a \frac{\sigma^a}{2}$ is a matrix.

Collect ψ-terms: $g\bar{\psi}\gamma^\mu \underbrace{\left[S^+ A'_\mu S + \frac{i}{g} S^+ \partial_\mu S\right]}_{\text{require } = A_\mu}\psi$

$$\Rightarrow\ A_\mu = S^+ A'_\mu S + \frac{i}{g} S^+ \partial_\mu S \ \Rightarrow\ S A_\mu S^+ = A'_\mu +$$

$$+\, \frac{i}{g}(\partial_\mu S)\, S^+ \Rightarrow \quad A'_\mu = S A_\mu S^+ - \frac{i}{g}(\partial_\mu S)\, S^+$$

$$\psi' = S\psi$$

non-abelian gauge transformation!

Def. Covariant derivative $D_\mu = \partial_\mu - i g A_\mu$

(note: now it's a matrix!)

$$\Rightarrow \quad \mathcal{L} = \bar{\psi}\,[i\gamma^\mu D_\mu - m]\,\psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

<u>But</u>: we never checked the invariance of $F^a_{\mu\nu} F^{a\mu\nu}$ term. What is $F^a_{\mu\nu}$ anyway? Using abelian analogy write $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]$

where $F_{\mu\nu} = F_{\mu\nu}^a \frac{\sigma^a}{2}$.

$$F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right] = \frac{i}{g}\left[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu\right] =$$

$$= \frac{i}{g}\left\{-ig\left[\partial_\mu, A_\nu\right] - ig\left[A_\mu, \partial_\nu\right] - g^2\left[A_\mu, A_\nu\right]\right\}$$

$$= \frac{i}{g}\left\{-ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - g^2\left[A_\mu, A_\nu\right]\right\} =$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$$

$$\Rightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$$

$$F_{\mu\nu}^a \frac{\tau^a}{2} = \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\frac{\sigma^a}{2} - ig A_\mu^b A_\nu^c \underbrace{\left[\frac{\sigma^b}{2}, \frac{\sigma^c}{2}\right]}_{i\varepsilon^{bca}\frac{\sigma^a}{2}} \leftarrow SU(2)$$

$$= \frac{\tau^a}{2}\left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,\varepsilon^{abc} A_\mu^b A_\nu^c\right]$$

$$\Rightarrow \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,\varepsilon^{abc} A_\mu^b A_\nu^c$$

~ true for SU(2)

~ other groups have different group structure constants:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,f^{abc} A_\mu^b A_\nu^c.$$

What happens to $F_{\mu\nu}$ under non-Abelian gauge transform?

Start with D_μ: $D_\mu = \partial_\mu - igA_\mu \rightarrow$

$$\rightarrow \partial_\mu - ig\left[S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S) S^{-1} \right] =$$

$$= S\left[\partial_\mu - igA_\mu \right] S^{-1} = S D_\mu S^{-1}$$

as $S \partial_\mu S^{-1} = \partial_\mu + S(\partial_\mu S^{-1})$

now: $\mathbb{1} = S S^{-1} \Rightarrow 0 = \partial_\mu (S S^{-1}) = (\partial_\mu S) S^{-1} + S(\partial_\mu S^{-1})$

$$\Rightarrow S(\partial_\mu S^{-1}) = -(\partial_\mu S) S^{-1} \Rightarrow S \partial_\mu S^{-1} = \partial_\mu - (\partial_\mu S) S^{-1}$$

$$\Rightarrow \boxed{D_\mu \rightarrow S D_\mu S^{-1}}$$

$$\Rightarrow F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] \rightarrow \frac{i}{g}[S D_\mu S^{-1}, S D_\nu S^{-1}]$$

$$= \frac{i}{g} S [D_\mu, D_\nu] S^{-1} = S F_{\mu\nu} S^{-1}$$

$$\Rightarrow \boxed{F_{\mu\nu} \rightarrow F'_{\mu\nu} = S F_{\mu\nu} S^{-1}}$$

$\Rightarrow$ Note that $F_{\mu\nu}$ is <u>not</u> invariant under gauge transformation if it is non-Abelian!

$$-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} \, tr\left(F_{\mu\nu} F^{\mu\nu}\right)$$

as $tr\left(\frac{\sigma^a}{2}\frac{\sigma^b}{2}\right) = \frac{1}{2}\delta^{ab}$ $\Rightarrow$ under non-abelian gauge transformation have

$$-\frac{1}{2} tr\left(F_{\mu\nu} F^{\mu\nu}\right) \to -\frac{1}{2} tr\left(F'_{\mu\nu} F'^{\mu\nu}\right) = -\frac{1}{2} tr\left[S F_{\mu\nu} \cancel{S^{-1}} \cancel{S} F^{\mu\nu} \cancel{S^{-1}}\right] = -\frac{1}{2} tr\left[F_{\mu\nu} F^{\mu\nu}\right]$$

$\Rightarrow$ the Lagrangian is invariant under non-abelian gauge transformation:

$$\boxed{\mathcal{L} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{2} tr\left(F_{\mu\nu} F^{\mu\nu}\right)}$$ true for any gauge group $SU(N)$

$$D_\mu = \partial_\mu - i g A_\mu \quad , \quad F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right].$$

The Higgs Mechanism (U(1) model)

~ Imagine a case when gauge symmetry is spontaneously broken

~ Goldstone Th'm does not apply: needs manifest Lorentz invariance (VEV is L. inv) & positivity of the norm. (to have G. boson state with $\langle u|u\rangle > 0$)

In gauge theories L. inv. gauges $\partial_\mu A^\mu = 0$ don't have > 0 of the norm, other gauges $A^0 = 0$, $\vec{\nabla}\cdot\vec{A} = 0$ are not manifestly $\mathcal{L}$.-inv..

Consider a Lagrangian:

$$\mathcal{L} = (D_\mu \varphi)^* (D_\mu \varphi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mu^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2$$

φ ~ complex scalar field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

A_μ ~ abelian gauge field, $D_\mu = \partial_\mu - i g A_\mu$.

The theory has a $U(1)$ local gauge symmetry:

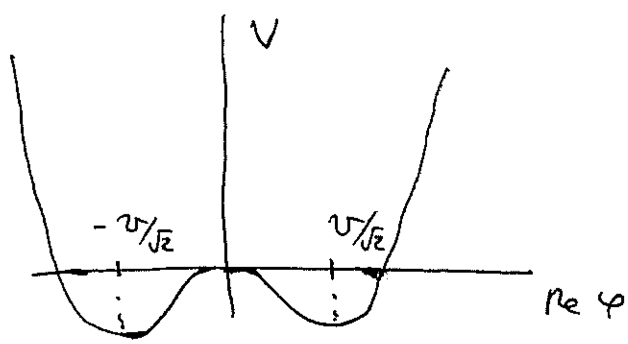
$$\begin{cases} \varphi \to \varphi' = e^{i\alpha(x)} \varphi \\ A_\mu \to A'_\mu = A_\mu + \frac{1}{g} \partial_\mu \alpha \end{cases}$$

The potential is $V(\varphi) = -\mu^2 \varphi^* \varphi + \lambda (\varphi^* \varphi)^2$.

$\Rightarrow$ the minimum is at

$$-2\mu^2 v' + 4 \lambda v'^3 = 0$$

$$\Rightarrow v' = \frac{\mu}{\sqrt{2\lambda}} \equiv \frac{v}{\sqrt{2}}$$



$\Rightarrow$ have an ∞ of vacua: $\langle 0|\varphi|0\rangle = v' e^{i\theta(x)}$, θ ~ real.

$\Rightarrow$ pick $\langle 0|\varphi|0\rangle = \frac{v}{\sqrt{2}} = v'$ as the vacuum.

$\Rightarrow$ SSB of gauge ~~the~~ symmetry!