Last time | The Electroweak Theory (cont'd)

Abelian theory: start with $\mathcal{L} = \bar{\psi}[i\partial\!\!\!/ - m]\psi$

It is invariant under global $U(1)$: $\psi(x) \to e^{i\alpha}\psi(x)$, $\alpha = $ real #.

Want local $U(1)$ symmetry: $\psi(x) \to e^{i\alpha(x)}\psi(x)$

$\Rightarrow$ arrive at

$$\boxed{\mathcal{L} = \bar{\psi}[iD\!\!\!\!/ - m]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

where we introduced a vector field $A_\mu(x)$

with
$$\boxed{\begin{aligned} D_\mu &\equiv \partial_\mu - igA_\mu \\ F_{\mu\nu} &= \frac{i}{g}[D_\mu, D_\nu] \end{aligned}}$$
~ covariant derivative; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Non-Abelian Theory: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ ~ spinor doublet

Start with $\mathcal{L} = \bar{\psi}[i\partial\!\!\!/ - m]\psi$.

This Lagrangian is invariant under $SU(2)$ global symmetry: $\psi(x) \to e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\psi(x)$.

Want local $SU(2)$ symmetry: $\psi(x) \to e^{i\vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2}}\psi(x)$.

arrive at $\boxed{\mathcal{L} = \bar{\psi}[i\not{D} - m]\psi - \frac{1}{2}\,\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}]}$

with the vector field(s) A_μ^a, $a = 1, 2, 3$ and

$A_\mu = \sum_{a=1}^{3} A_\mu^a \frac{\sigma^a}{2}$ ~ 2×2 matrix (Hermitean, traceless)

The covariant derivative is

$$\boxed{D_\mu = \partial_\mu - ig A_\mu}$$ ~ also a matrix now

and the field strength tensor is

$$\boxed{F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]}$$

In components: $\boxed{F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,\varepsilon^{abc} A_\mu^b A_\nu^c}$

$\left(F_{\mu\nu} = F_{\mu\nu}^a \frac{\sigma^a}{2}\right)$

SU(2) local:

$S = e^{i\vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2}}$

$$\boxed{\begin{aligned} &\psi \to \psi' = S\psi \\ &A_\mu \to A'_\mu = S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1} \end{aligned}}$$

$\boxed{D_\mu \to S D_\mu S^{-1}}$ and $\boxed{F_{\mu\nu} \to S F_{\mu\nu} S^{-1}}$.

Generalization: SU(N) Gauge Theory

For a SU(N) gauge theory use $N \times N$ generators of su(N) in the fundamental representation t^a:

$$[t^a, t^b] = i f^{abc} t^c$$

$\uparrow$ structure constants

$\Rightarrow$ $A_\mu = A_\mu^a t^a$, $a = 1, \ldots, N^2 - 1$

$F_{\mu\nu} = F_{\mu\nu}^a t^a$ $\Rightarrow$ again $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]$ with the covariant derivative $D_\mu = \partial_\mu - ig A_\mu$. One can show that

$$\boxed{F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c}$$

The gauge-invariant Lagragian is then:

$$\boxed{\mathcal{L} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}$$

ψ^i, $i = 1, \ldots, N$ ~ N different spinors

$A_\mu' = S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S) S^{-1}$, $\psi' = S\psi$ ~ gauge transform

$D_\mu \to S D_\mu S^{-1}$

$\Rightarrow$ $\mathcal{L}$ is invariant under SU(N) local gauge symmetry!

$$-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$$

as $\mathrm{tr}\left(\frac{\sigma^a}{2} \frac{\sigma^b}{2}\right) = \frac{1}{2}\delta^{ab}$ => under non-abelian gauge transformation have

$$-\frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) \to -\frac{1}{2}\mathrm{tr}\left(F'_{\mu\nu}F'^{\mu\nu}\right) = -\frac{1}{2}\mathrm{tr}\left[\cancel{S}\, F_{\mu\nu} \cancel{S^{-1}}\, \cancel{S}\, F^{\mu\nu} \cancel{S^{-1}}\right] = -\frac{1}{2}\mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right]$$

=> the Lagrangian is invariant under non-Abelian gauge transformation:

$$\boxed{\mathcal{L} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right)}$$ true for any gauge group $SU(N)$

$$D_\mu = \partial_\mu - i g A_\mu \quad , \quad F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right].$$

The Higgs Mechanism (U(1) model)

~ Imagine a case when gauge symmetry is spontaneously broken

~ Goldstone Th'm does not apply: needs manifest Lorentz invariance (VEV is L. inv) & positivity of the norm. (to have G. boson state with $\langle u|u\rangle > 0$)

In gauge theories L. inv. gauges $\partial_\mu A^\mu = 0$ don't have > 0 of the norm, other gauges $A^0 = 0$, $\vec{\nabla}\cdot\vec{A} = 0$ are not manifestly $\mathcal{L}$.-inv..

Consider the Lagrangian:

$$\mathcal{L} = \left(D_\mu \varphi\right)^* \left(D^\mu \varphi\right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mu^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2$$

$\varphi \sim$ complex scalar field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$A_\mu \sim$ abelian gauge field, $D_\mu = \partial_\mu - ig A_\mu$.

The theory has a $U(1)$ local gauge symmetry:

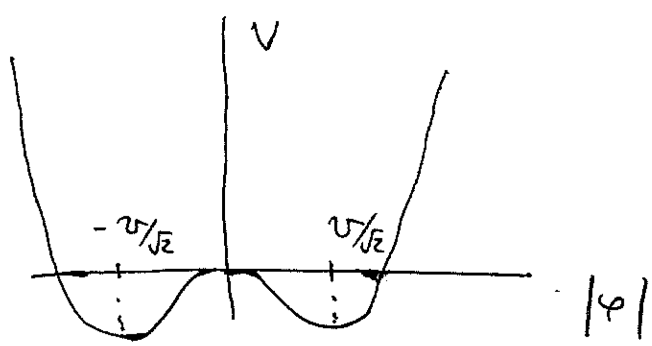
$$\begin{cases} \varphi \to \varphi' = e^{i\alpha(x)} \varphi \\ A_\mu \to A'_\mu = A_\mu + \frac{1}{g} \partial_\mu \alpha \end{cases}$$

The potential is $V(\varphi) = -\mu^2 \varphi^* \varphi + \lambda (\varphi^* \varphi)^2$.

$\Rightarrow$ the minimum is at $|\varphi| = v'$:

$$-2\mu^2 v' + 4\lambda v'^3 = 0$$

$$\Rightarrow v' = \frac{\mu}{\sqrt{2\lambda}} \equiv \frac{v}{\sqrt{2}}$$



$\Rightarrow$ have an ∞ of vacua: $\langle 0 | \varphi | 0 \rangle = v' e^{i\theta(x)}$, $\theta \sim$ real.

$\Rightarrow$ pick $\langle 0 | \varphi | 0 \rangle = \frac{v}{\sqrt{2}} = v'$ as the vacuum.

$\Rightarrow$ SSB of gauge symmetry!

Write $\varphi = \frac{\rho'(x)}{\sqrt{2}} e^{i\theta(x)}$ with $\rho'(x), \theta(x)$ real fields. (140)

$$\mathcal{L} = \left[\left(\partial_\mu + ig A_\mu\right) \frac{\rho'}{\sqrt{2}} e^{-i\theta}\right] \cdot \left[\left(\partial^\mu - ig A^\mu\right) \frac{\rho'}{\sqrt{2}} e^{i\theta}\right] -$$

$$- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\mu^2}{2}(\rho')^2 - \frac{\lambda}{4}(\rho')^4 = \left[\left(\partial_\mu + ig A_\mu - i\partial_\mu\theta\right) \cdot \frac{\rho'}{\sqrt{2}}\right]$$

$$\cdot \left[\left(\partial^\mu - ig A^\mu + i\partial^\mu\theta\right)\frac{\rho'}{\sqrt{2}}\right] - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\mu^2}{2}(\rho')^2 - \frac{\lambda}{4}(\rho')^4$$

$\Rightarrow$ define $B_\mu \equiv A_\mu - \frac{1}{g}\partial_\mu\theta$ ($F_{\mu\nu}$ remains the same ~ like a gauge transf.)

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial_\mu + ig B_\mu\right)\rho'\right]\left[\left(\partial^\mu - ig B^\mu\right)\rho'\right] - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{\mu^2}{2}(\rho')^2 - \frac{\lambda}{4}(\rho')^4.$$

with $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = F_{\mu\nu}$

Now, in the new vacuum $\langle 0|\rho'|0\rangle = v$, $\langle 0|\theta|0\rangle = 0$

$\Rightarrow$ write $\rho' = \rho + v$ $\Rightarrow$ get

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial_\mu + ig B_\mu\right)(\rho + v)\right]\left[\left(\partial^\mu - ig B^\mu\right)(\rho + v)\right] - \frac{1}{4} G_{\mu\nu} G^{\mu\nu}$$

$$+ \frac{\mu^2}{2}(\rho + v)^2 - \frac{\lambda}{4}(\rho + v)^4.$$

kinetic term $= \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \frac{1}{2} g^2 B_\mu B^\mu (\rho + v)^2 +$

$$+ \frac{1}{2}\cancel{\left(ig B_\mu(\rho + v)\partial^\mu\rho - ig B^\mu(\rho + v)\partial_\mu\rho\right)}^{\,0} = \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho$$

$$+ \frac{1}{2} g^2 B_\mu B^\mu (\rho + v)^2.$$

potential: $\frac{\mu^2}{2}\left(2\rho\cdot v+\rho^2\right)-\frac{\lambda}{4}\left(\rho^4+4\rho^3 v+6\rho^2 v^2\right.$
$\left.+4\rho v^3+v^4\right)=\rho\,\cancel{\left(\mu^2 v-\lambda v^3\right)}\to 0+\rho^2\cdot\left(\frac{\mu^2}{2}-\right.$
$\left.-\frac{3}{2}\underbrace{\lambda v^2}_{\mu^2}\right)-\lambda\rho^3 v-\frac{\lambda}{4}\rho^4=-\mu^2\rho^2-\lambda\rho^3 v-\frac{\lambda}{4}\rho^4$

$$\boxed{\mathcal{L}=\frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho-\mu^2\rho^2-\frac{1}{4}G_{\mu\nu}G^{\mu\nu}+\frac{1}{2}g^2v^2B_\mu B^\mu+\frac{1}{2}g^2B_\mu B^\mu\left(2\rho v+\rho^2\right)-\lambda\rho^3 v-\frac{\lambda}{4}\rho^4}$$

particle content:

~ a scalar ρ with mass $\mu\sqrt{2}$.

~ a massive gauge field B_μ with mass
$m_B=g v$ (see HW #2, problem on Proca Lagrangian: $\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{m^2}{2}A_\mu A^\mu$.)

~ field θ got "eaten up" by B_μ, as massless gauge field A_μ had 2 d.o.f., now $\oplus$ 1 (θ)

$\Rightarrow$ B_μ has 3 degrees of freedom.

θ ~ "would-be" Goldstone boson

~ if we had not absorbed θ into B_μ would have gotten terms like $A_\mu\partial^\mu\theta$ ~ not clear how to interpret.
(related to negative norm problem) (---------) ?

⇒ SSB of gauge symmetry ~ no Goldstone bosons

~ but get massive vector fields!

(e.g. Meissner effect in superconductivity when photon gets a "mass" and is screened in superconductor ~ P.W. Anderson, '58)

⇒ in particle physics this is known as the Higgs phenomenon. (P. Higgs, 1964).

SU(2) ⊗ U(1) Electroweak Theory.

history: Pauli postulated neutrino to explain β-decay (1930).

Fermi ('34) : to explain β-decay $n \to p e \bar{\nu}$

suggested an interaction term

$$\mathcal{L}_F = -\frac{G_F}{\sqrt{2}} [\bar{p} \gamma_\mu n][\bar{e} \gamma^\mu \nu] + h.c.$$

with $G_F = \frac{10^{-5}}{m_p^2}$. ⇒ but as $[G_F] = \frac{1}{M^2}$ ⇒ not renormalizable

⇒ as theory has W, Z vector bosons ⇒ Glashow, Salam proposed a gauge theory ('61, '64)