Last time | Interactions of W's and Z's with Quarks (cont'd)

$$\mathcal{U}_L = \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L, \quad D_L = \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L, \quad \psi = \begin{pmatrix} \mathcal{U}_L \\ D_L \end{pmatrix}$$

$$M = \begin{pmatrix} \mathbb{1}_{3\times 3} & 0 \\ 0 & V \end{pmatrix}, \quad V = \text{CKM matrix}$$

$$\boxed{\begin{aligned} \mathcal{L}_{\text{quarks+gauge}} &= \bar{\psi}\, i\gamma^\mu \Big[\partial_\mu - i\frac{g'}{6}\left(A_\mu \cos\theta_W - Z_\mu \sin\theta_W\right) \\ &- ig\frac{\tau^3}{2}\left(A_\mu \sin\theta_W + Z_\mu \cos\theta_W\right) - i\frac{g}{\sqrt{2}} M^\dagger\left(\tau^+ W_\mu + \tau^- W_\mu^\dagger\right) M \Big]\psi \\ &+ \sum_f \bar{R}_f\, i\gamma^\mu \left(\partial_\mu - i\frac{g'}{2} Y_f \left(A_\mu \cos\theta_W - Z_\mu \sin\theta_W\right)\right) R_f \end{aligned}}$$

$$\tau^\pm = \frac{\tau^1 \pm i\tau^2}{2}.$$

(i) Derived the charged current ~ coupling of W to quarks:

$$\boxed{\begin{aligned} \mathcal{L}_{c.c.} = \frac{g}{2\sqrt{2}} \Big\{ &\bar{u}\,\gamma\cdot W(1-\gamma_5)\left[V_{ud}\, d + V_{us}\, s + V_{ub}\, b\right] \\ + &\bar{c}\,\gamma\cdot W(1-\gamma_5)\left[V_{cd}\, d + V_{cs}\, s + V_{cb}\, b\right] \\ + &\bar{t}\,\gamma\cdot W(1-\gamma_5)\left[V_{td}\, d + V_{ts}\, s + V_{tb}\, b\right] \Big\} + h.c. \end{aligned}}$$

(ii) Neutral current: coupling of Z & γ to quarks

$$\boxed{\mathcal{L}_{\text{photons}} = \sum_f e_f \, \bar{q}_f \, \gamma \cdot A \, q_f} \quad \text{as expected!}$$

$e = g' \cos \theta_W = g \sin \theta_W$ used this.

$$\mathcal{L}^{photons} = \frac{g'}{6}\cos\theta_W\, \bar{\Psi}\, \gamma\cdot A\, \Psi + \frac{g}{2}\sin\theta_W\, \bar{\Psi}\, \gamma\cdot A\, \tau^3 \Psi$$

$$+ \sum_f \frac{g'}{2} Y_f \cos\theta_W\, \bar{R}_f\, \gamma\cdot A\, R_f$$

$\Rightarrow$ remember $e = g'\cos\theta_W = g\sin\theta_W$

$$\mathcal{L}^{photons} = \bar{\Psi}\, \gamma\cdot A \left(\frac{e}{6} + \frac{e}{2}\tau^3\right)\Psi + \sum_f \frac{e}{2} Y_f\, \bar{R}_f\, \gamma\cdot A\, R_f$$

$$\Rightarrow e\left(\frac{1}{6} + \frac{\tau^3}{2}\right) = e\left(\frac{Y}{2} + \frac{\tau^3}{2}\right) = e\left(\frac{Y}{2} + I_3\right) = Q_{LHQ} = e_f$$

$\Rightarrow$ Gell-Mann-Nishijima formula

(check: $\frac{Y}{2} + \frac{\tau^3}{2} = \frac{1}{6} + \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}\frac{2}{3} & 0\\ 0 & -\frac{1}{3}\end{pmatrix}$ ← charge of u, c, t; ← charge of d, s, b)

$$\frac{e}{2}\cdot Y_f = e\cdot\frac{1}{2}\cdot\begin{cases} 4/3 & \text{for } u, c, t\\ -2/3 & \text{for } d, s, b\end{cases} = e\cdot\begin{cases} 2/3 & \text{for } u, c, t\\ -1/3 & \text{for } d, s, b\end{cases}$$

$\Rightarrow$ get $Q_{RHQ} = e_f$ again

$$\boxed{e_f = e\cdot\begin{cases} 2/3 & \text{for } u, c, t\\ -1/3 & \text{for } d, s, b\end{cases}}$$

$$\Rightarrow \boxed{\mathcal{L}^{photons} = \sum_f e_f\, \bar{q}_f\, \gamma\cdot A\, q_f}$$

Regular QED term as expected!

b) Z - bosons: $\mathcal{L}^{Z} = \bar{\Psi}\gamma^\mu\left[-\frac{g'}{6}\sin\theta_W + g\frac{\tau^3}{2}\cdot\right.$

$\left.\cdot\cos\theta_W\right] Z_\mu \Psi - \sum_f \bar{R}_f \gamma^\mu Z_\mu \frac{g'}{2} Y_f \sin\theta_W R_f = \left| \begin{array}{l} g'\sin\theta_W = \\ = g\frac{\sin^2\theta_W}{\cos\theta_W}\end{array}\right.$

$$= (\bar{U}_L \ \bar{D}_L)\,\gamma\cdot Z \begin{pmatrix} \frac{g}{2}\cos\theta_W - \frac{g}{6}\frac{\sin^2\theta_W}{\cos\theta_W} & 0 \\ 0 & -\frac{g}{2}\cos\theta_W - \frac{g}{6}\frac{\sin^2\theta_W}{\cos\theta_W} \end{pmatrix} \begin{pmatrix} U_L \\ D_L \end{pmatrix}$$

$$- \frac{g\sin^2\theta_W}{2\cos\theta_W}\cdot\sum_f \bar{R}_f\gamma\cdot Z\, Y_f R_f = \frac{g}{2\cos\theta_W}\Big[\bar{u}_L\gamma\cdot Z\, u_L\cdot$$

$$\cdot\left(\cos^2\theta_W - \frac{1}{3}\sin^2\theta_W\right) - \bar{D}_L\gamma\cdot Z\, D_L\left(\cos^2\theta_W + \frac{1}{3}\sin^2\theta_W\right) -$$

$$- \frac{g\sin^2\theta_W}{2\cos\theta_W}\left(\bar{u}_R\gamma\cdot Z\, u_R\cdot\frac{4}{3} + \bar{D}_R\gamma\cdot Z\, D_R\left(-\frac{2}{3}\right)\right)\Big]$$

$$\Rightarrow \mathcal{L}^{Z} = \frac{g}{4\cos\theta_W}\Big\{\bar{u}\,\gamma\cdot Z\left[(1-\gamma_5)\left(1-\frac{4}{3}\sin^2\theta_W\right) - (1+\gamma_5)\frac{4}{3}\sin^2\theta_W\right]\cdot$$

$$\cdot u - \bar{D}\,\gamma\cdot Z\left[(1-\gamma_5)\left(1-\frac{2}{3}\sin^2\theta_W\right) - (1+\gamma_5)\frac{2}{3}\sin^2\theta_W\right] D$$

Putting photons & Z-bosons together get

$$\mathcal{L}_{nc.} = \frac{g}{4\cos\theta_W}\Big\{\bar{u}\,\gamma\cdot Z\left[(1-\gamma_5)\left(1-\frac{4}{3}\sin^2\theta_W\right) - (1+\gamma_5)\frac{4}{3}\sin^2\theta_W\right]u -$$

$$- \bar{D}\,\gamma\cdot Z\left[(1-\gamma_5)\left(1-\frac{2}{3}\sin^2\theta_W\right) - (1+\gamma_5)\frac{2}{3}\sin^2\theta_W\right]D\Big\} + \sum_f e_f\,\bar{q}_f\,\gamma\cdot A\, q_f$$

(neutral current $\mathcal{L}$)

Feynman Rules and Cross Sections

⇒ to perform calculations in the interacting field theory need to construct perturbation theory using Feynman rules.

⇒ non-perturbative methods exist (e.g. lattice, effective theories), but often difficult and sometimes impossible to apply to high energy scattering

⇒ as $\alpha_{EM} = \frac{e^2}{4\pi} \simeq \frac{1}{137} \ll 1$, $\frac{g^2}{4\pi} \simeq \frac{1}{30} \ll 1$ ⇒

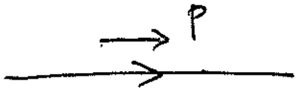
⇒ can construct pert. theory in them!

Consider QED:
$$\mathcal{L}_{QED} = \bar{\psi}\, i\gamma\cdot\partial\,\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\,\bar{\psi}\,\gamma\cdot A\,\psi$$

a theory of electrons ψ & photons A_μ.

Feynman Rules for QED

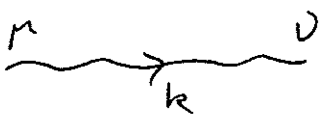
electron propagator



$$\frac{i}{\not{p} - m + i\varepsilon} = \frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}$$ (assigned to each internal electron line)

$$\not{p} = \gamma^\mu p_\mu .$$

photon propagator

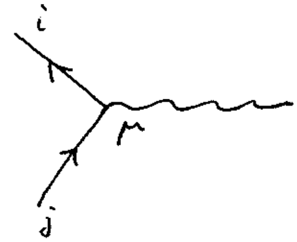


$$\frac{-i}{k^2 + i\varepsilon}\left[g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right],$$ $\xi = 1$ Feynman gauge, $\xi = 0$ Landau gauge

(assigned to each internal photon line)

- electron-photon vertex:

$-ie\gamma^\mu$



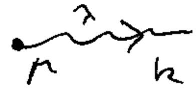
(in more complicated theory get other factors on top of γ^μ)

External photon lines:

incoming:

$\Rightarrow \varepsilon_\mu^\lambda(k)$, λ = polarization

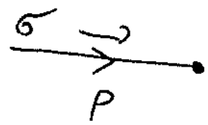
out going:



$\Rightarrow \varepsilon_\mu^{\lambda *}(k)$

External fermions:

incoming:



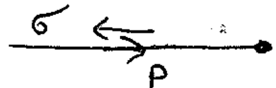
$\Rightarrow u_\sigma(p)$

outgoing:

$\Rightarrow \bar{u}_\sigma(p)$

External anti-fermions:

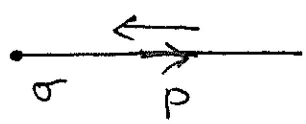
incoming:



$\Rightarrow \bar{v}_\sigma(p)$

(note that momentum flows in, fermion # flows out)

outgoing:



$\Rightarrow v_\sigma(p)$

Cross section = $\dfrac{\text{event prob. per unit volume \& time}}{(\text{incident flux}) \times (\text{target density})}$

$\underbrace{|\text{final state}\rangle}_{=|\psi_f\rangle} = S\ \underbrace{|\text{initial state}\rangle}_{=|\psi_i\rangle}$

Def. $\longrightarrow$ S - matrix (time-evolution operator)

$$|\psi_f\rangle = S|\psi_i\rangle = \left[1 + (S-1)\right]|\psi_i\rangle = |\psi_i\rangle + $$
$$+ (S-1)|\psi_i\rangle$$

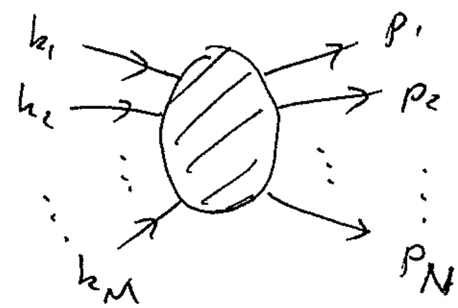
$\Rightarrow$ the interaction is in $S-1$ $\Rightarrow$

Def. T-matrix is defined by $\boxed{S = 1 + i\,T}$

Unitarity: $S^+ S = 1 \Rightarrow 1 - iT^+ + iT + T^+T = 1$

$\Rightarrow i(T - T^+) = -T^+T \Rightarrow 2\,\mathrm{Im}\,T = T^+T \Rightarrow$ optical theorem

Def. Scattering amplitude:



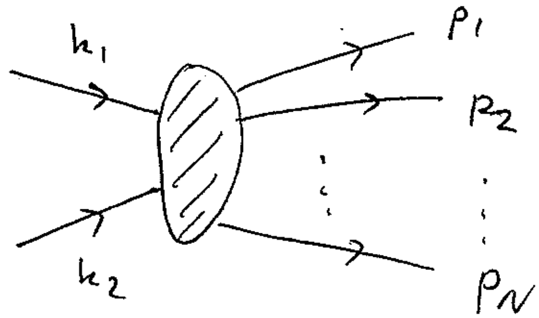
$\Rightarrow$ initial state is $|k_1, k_2, \ldots, k_M\rangle$

final (outgoing) free-particle state $|p_1, \ldots, p_N\rangle$

$\Rightarrow$ transition amplitude M is defined by

$$\langle p_1, \ldots, p_N | S - 1 | k_1, \ldots, k_M\rangle = i\langle p_1, \ldots, p_N | T | k_1, \ldots, k_M\rangle \equiv$$
$$\equiv (2\pi)^4 \delta^{(4)}(p_1 + \ldots + p_N - k_1 - \ldots - k_M)\; i\, M(k_1, \ldots, k_M; p_1, \ldots, p_N)$$

For simplicity consider $2 \to N$ process:



$\Rightarrow$ one can show that the <u>cross section</u> for the process is:

$$\boxed{d\sigma = \frac{1}{2\mathcal{E}_{k_1}\, 2\mathcal{E}_{k_2}\, |\vec{v}_1 - \vec{v}_2|} \prod_{i=1}^{N} \frac{d^3 p_i}{(2\pi)^3\, 2\mathcal{E}_i} \left| M(k_1, k_2; p_1, \ldots, p_N) \right|^2 \cdot (2\pi)^4\, \delta^{(4)}\Big(k_1 + k_2 - \sum_{j=1}^{N} p_j\Big)}$$

(see Peskin & Schroeder, Ryder, ...)

$\vec{v}_1, \vec{v}_2$ ~ 3-velocities of the incoming particles.

Is this object Lorentz-invariant?

$$\frac{d^3 p_i}{(2\pi)^3\, 2\mathcal{E}_i} = \int \frac{d^4 p_i}{(2\pi)^4}\, 2\pi\, \delta(p_i^2 - m_i^2)\, \theta(p_i^0) \sim \text{Lorentz-inv.}$$

$\delta^{(4)}$ is $\mathcal{L}$-inv. (why?)

$|M|^2$ is $\mathcal{L}$ inv. (see definition)

What about $\mathcal{E}_{k_1} \mathcal{E}_{k_2} |\vec{v}_1 - \vec{v}_2|$? Note that $v_i = \frac{\vec{k}_i}{\mathcal{E}_{k_i}}$

$$\Rightarrow \mathcal{E}_{k_1} \mathcal{E}_{k_2} |\vec{v}_1 - \vec{v}_2| = \mathcal{E}_{k_1} \mathcal{E}_{k_2} \left| \frac{\vec{k}_1}{\mathcal{E}_{k_1}} - \frac{\vec{k}_2}{\mathcal{E}_{k_2}} \right| = \left| \mathcal{E}_{k_2} \vec{k}_1 - \mathcal{E}_{k_1} \vec{k}_2 \right|$$

$$= \sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2} \Rightarrow \text{Lorentz-invariant.}$$

($k_1 \cdot k_2 = \mathcal{E}_1 \mathcal{E}_2 - \vec{k}_1 \cdot \vec{k}_2 \Rightarrow \sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2} = \left((\mathcal{E}_1 \mathcal{E}_2 - \vec{k}_1 \cdot \vec{k}_2)^2 - m_1^2 m_2^2 \right)^{1/2} = \left(\mathcal{E}_1^2 \mathcal{E}_2^2 + (\vec{k}_1 \cdot \vec{k}_2)^2 - 2 \mathcal{E}_1 \mathcal{E}_2 \vec{k}_1 \cdot \vec{k}_2 - m_1^2 m_2^2 \right)^{1/2}$

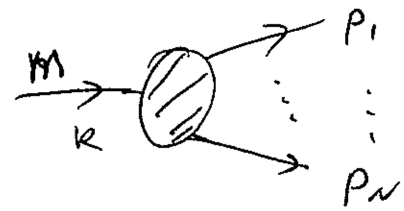
$= \underset{\vec{k}_1 \parallel \vec{k}_2}{(\text{collinear case})} = \left(\mathcal{E}_1^2 \mathcal{E}_2^2 + (\mathcal{E}_1^2 - m_1^2)(\mathcal{E}_2^2 - m_2^2) - 2 \mathcal{E}_1 \mathcal{E}_2 \vec{k}_1 \cdot \vec{k}_2 - m_1^2 m_2^2 \right)^{1/2} = \left(2 \mathcal{E}_1^2 \mathcal{E}_2^2 - m_1^2 \mathcal{E}_2^2 - m_2^2 \mathcal{E}_1^2 - 2 \mathcal{E}_1 \mathcal{E}_2 \vec{k}_1 \cdot \vec{k}_2 \right)^{1/2} = \left(\mathcal{E}_2^2 \vec{k}_1^{\,2} + \mathcal{E}_1^2 \vec{k}_2^{\,2} - 2 \mathcal{E}_1 \mathcal{E}_2 \vec{k}_1 \cdot \vec{k}_2 \right)^{1/2}$

$= | \mathcal{E}_2 \vec{k}_1 - \mathcal{E}_1 \vec{k}_2 |$)

$\Rightarrow \mathcal{E}_{k_1} \mathcal{E}_{k_2} | \vec{v}_1 - \vec{v}_2 |$ is Lorentz invariant iff $\vec{v}_1, \vec{v}_2, \mathcal{E}_{k_1}, \mathcal{E}_{k_2}$ are taken in a frame where $\vec{v}_1 \parallel \vec{v}_2$!

<u>Decay rate</u>: imagine one particle decaying into N particles:

work in particle's rest frame



$\Rightarrow$ start from the x-section:

$$\frac{1}{2\mathcal{E}_{k_1} 2\mathcal{E}_{k_2} |\vec{v}_1 - \vec{v}_2|} \longrightarrow \frac{1}{2M}, \quad m = \text{particle mass}$$

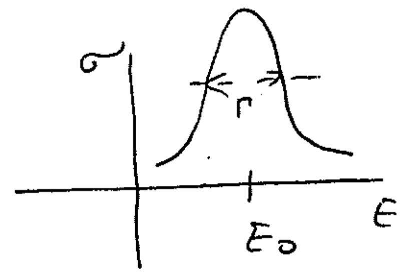
$$\boxed{d\Gamma = \frac{1}{2m} \prod_{i=1}^{N} \frac{d^3 p_i}{(2\pi)^3\, 2E_i} \,|M(k; p_1, \ldots, p_N)|^2 (2\pi)^4 \delta\Big(k - \sum_{j=1}^{N} p_j\Big)} \qquad (17.6)$$

~ decay rate

$$\Gamma = \frac{\#\text{ decays per unit time}}{\#\text{ of particles}}$$

Breit-Wigner formula for scattering amplitude:

$$f(E) \sim \frac{1}{E - E_0 + i\,\Gamma/2}$$



$$\Rightarrow \sigma(E) \propto |f(E)|^2 \sim \frac{1}{(E-E_0)^2 + \frac{\Gamma^2}{4}}$$

width of resonance peak!

$\Rightarrow$ NB in calculating $|M|^2$ sum over spins, etc in the final state, average over them in the initial state.

Decay of the Z-boson

$\Rightarrow$ let's calculate decay rate of the Z-boson

$\Rightarrow$ the Z-boson interaction Lagrangian is:

$$\mathcal{L}_Z = \frac{g}{4\cos\theta_W}\Big\{ \bar{\nu}_e \gamma\cdot Z(1-\gamma_5)\nu_e + 2\sin^2\theta_W\, \bar{e}\,\gamma\cdot Z(1+\gamma_5)e +$$

$$+(2\sin^2\theta_W - 1)\,\bar{e}\,\gamma\cdot Z(1-\gamma_5)e \overset{+(\mu,\tau)}{+} \bar{u}\,\gamma\cdot Z\Big[(1-\gamma_5)\Big(1-\tfrac{4}{3}\sin^2\theta_W\Big) - (1+\gamma_5)$$

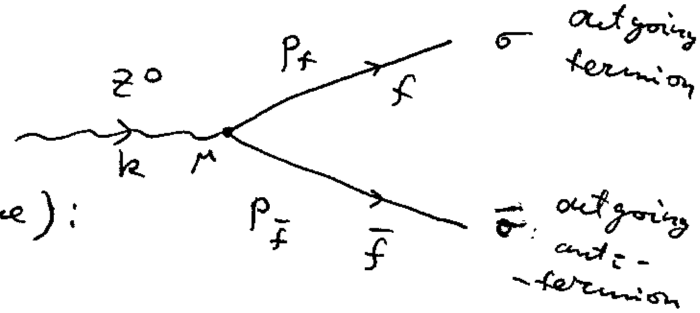
$$\cdot\tfrac{4}{3}\sin^2\theta_W\Big]u - \bar{D}\,\gamma\cdot Z\Big[(1-\gamma_5)\Big(1-\tfrac{2}{3}\sin^2\theta_W\Big) - (1+\gamma_5)\tfrac{2}{3}\sin^2\theta_W\Big]D\Big\}$$

$Z \to W^+W^-$, $Z \to \gamma Z$, $Z \to W^+W^- Z, \ldots$ are all prohibited by energy conservation! $M_Z \simeq 91\,\text{GeV} < 2M_W = 2\cdot 80\,\text{GeV} \ldots$

For each fermion species the Lagrangian looks like: $\mathcal{L} = \bar{\Psi}\, \gamma \cdot Z \left[a_f (1-\gamma_5) - b_f (1+\gamma_5) \right] \Psi$

with a_f, b_f coefficients being species-dependent.

Consider Z-decay:



Amplitude (in Z^0 rest frame):

$$M = \underbrace{-i}_{\text{vertex}} \; \bar{u}_\sigma(p_f) \underbrace{\gamma^\mu \left[a_f (1-\gamma_5) - b_f (1+\gamma_5) \right]}_{\text{vertex}} v_{\bar\sigma}(p_{\bar f}) \cdot \varepsilon^\lambda_\mu (k)$$

Need to find:

$$\frac{1}{3} \sum_{\lambda, \sigma, \bar\sigma} |M|^2 = \frac{1}{3} \sum_{\lambda,\sigma,\bar\sigma} \bar{u}_\sigma\, \gamma \cdot \varepsilon^\lambda \left[a_f (1-\gamma_5) - b_f (1+\gamma_5) \right]$$

$\lambda, \sigma, \bar\sigma$ ← sum over spins

↑ average over initial polarizations

$$\cdot\, v_{\bar\sigma} \; \underbrace{\bar{v}_{\bar\sigma} \gamma^0}_{v^+ \underbrace{\gamma^0\gamma^0}_{1}} \left[a_f^* (1-\underbrace{\gamma_5^+}_{=\gamma_5}) - b_f^* (1+\underbrace{\gamma_5^+}_{=\gamma_5}) \right] \gamma^+ \cdot \varepsilon^{\lambda *} \gamma^0 u_\sigma$$

$\Rightarrow$ as $(\gamma^0)^+ = \gamma^0$, $(\gamma^i)^+ = -\gamma^i$, $i = 1,2,3$

$$\Rightarrow \gamma^0 \gamma_\mu^+ \gamma^0 = \begin{cases} \gamma^0 & \text{if } \mu = 0 \\ \gamma^i & \text{if } \mu = i \end{cases} \Rightarrow \gamma^0 (\gamma^\mu)^+ \gamma^0 = \gamma^\mu$$

as $\{\gamma^\mu, \gamma^5\} = 0 \Rightarrow \gamma^0 \gamma^5 (\gamma^\mu)^+ \gamma^0 = \begin{cases} -\gamma^5\gamma^0, & \mu = 0 \\ -\gamma^5 \gamma^i, & \mu = i \end{cases}$

$\Rightarrow \gamma^5 \to -\gamma^5$