**Last time** We constructed the full neutral current Lagrangian in the quarks + gauge bosons sector of the standard model:

$$\boxed{\begin{aligned}\mathcal{L}_{n.c.} = \frac{g}{4\cos\theta_W}\Big\{&\bar{u}\,\gamma\cdot Z\left[(1-\gamma_5)\left(1-\tfrac{4}{3}\sin^2\theta_W\right)-(1+\gamma_5)\tfrac{4}{3}\sin^2\theta_W\right]u \\ &-\bar{D}\,\gamma\cdot Z\left[(1-\gamma_5)\left(1-\tfrac{2}{3}\sin^2\theta_W\right)-(1+\gamma_5)\tfrac{2}{3}\sin^2\theta_W\right]D\Big\}+\sum_f e_f\,\bar{q}_f\,\gamma\cdot A\,q_f\end{aligned}}$$

## Feynman Rules and Cross Sections

We reviewed Feynman rules for QCD & the expressions for the cross sections

$$d\sigma = \frac{1}{2E_{k_1}\,2E_{k_2}\,|\vec{v}_1-\vec{v}_2|}\prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3\,2E_{p_i}}\;\underbrace{\frac{1}{n!}}_{\text{if particles are identical}}\;|M(k_1,k_2;p_1,\dots,p_n)|^2\cdot(2\pi)^4\,\delta^4\Big(k_1+k_2-\sum_{j=1}^{n}p_j\Big)$$

and for the decay rates

$$d\Gamma = \frac{1}{2m}\,\frac{1}{n!}\prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3\,2E_{p_i}}\;|M(k;p_1,\dots,p_n)|^2\cdot(2\pi)^4\,\delta^4\Big(k-\sum_{j=1}^{n}p_j\Big).$$

$M$ = scattering amplitude, in $|M|^2$ sum over final state quantum #'s, average ... rate.

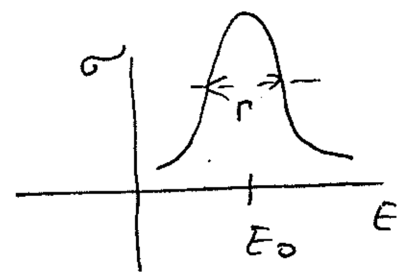
$$d\Gamma = \frac{1}{2m} \prod_{i=1}^{N} \frac{d^3 p_i}{(2\pi)^3 2\mathcal{E}_i} \, |M(k; p_1, \dots, p_N)|^2 (2\pi)^4 \delta\Big(k - \sum_{j=1}^{N} p_j\Big)$$

~ decay rate

$$\Gamma = \frac{\#\ \text{decays per unit time}}{\#\ \text{of particles}}$$

Breit-Wigner formula for scattering amplitude:

$$f(E) \sim \frac{1}{E - E_0 + i\,\Gamma/2}$$




$$\Rightarrow \sigma(E) \propto |f(E)|^2 \sim \frac{1}{(E-E_0)^2 + \frac{\Gamma^2}{4}}$$

Width of resonance peak!

⇒ NB in calculating $|M|^2$ sum over spins, etc in the final state, average over them in the initial state.

## Decay of the Z-boson

⇒ let's calculate decay rate of the Z-boson

⇒ the Z-boson interaction Lagrangian is:

$$\mathcal{L}_Z = \frac{g}{4\cos\theta_W} \Big\{ \bar{\nu}_e \gamma\cdot Z (1-\gamma_5) \nu_e + 2\sin^2\theta_W \, \bar{e} \gamma\cdot Z (1+\gamma_5) e +$$

$$+ (2\sin^2\theta_W - 1) \bar{e} \gamma\cdot Z (1-\gamma_5) e \overset{+(\mu,\tau)}{+} \bar{U} \gamma\cdot Z \Big[ (1-\gamma_5)\Big(1 - \tfrac{4}{3}\sin^2\theta_W\Big) - (1+\gamma_5)$$

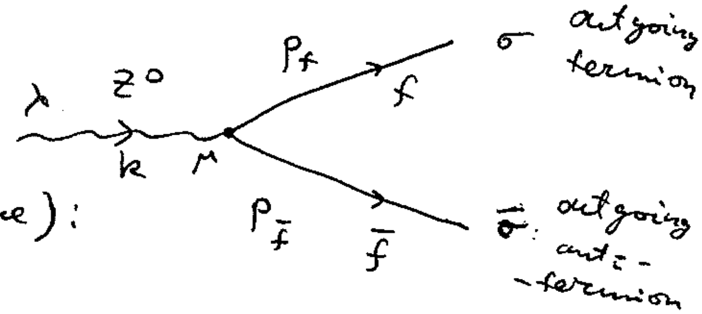
$$\cdot \tfrac{4}{3}\sin^2\theta_W \Big] U - \bar{D} \gamma\cdot Z \Big[ (1-\gamma_5)\Big(1 - \tfrac{2}{3}\sin^2\theta_W\Big) - (1+\gamma_5)\tfrac{2}{3}\sin^2\theta_W \Big] D \Big\}$$

$Z \to W^+W^-$, $Z \to \gamma Z$, $Z \to W^+W^- Z, \dots$ are all prohibited by energy conservation! $M_Z \simeq 91\,\text{GeV} < 2M_W = 2\cdot 80\,\text{GeV} \dots$

For each fermion species the Lagrangian looks like: $\mathcal{L} = \bar{\Psi}\, \gamma \cdot Z \left[ a_f (1-\gamma_5) - b_f (1+\gamma_5) \right] \Psi$

with $a_f, b_f$ coefficients being species-dependent.

Consider Z-decay:




Amplitude (in $Z^0$ rest frame):

$$i\mathcal{M} = \underbrace{i}_{\text{vertex}} \; \bar{u}_\sigma(p_f) \underbrace{\gamma^\mu \left[ a_f (1-\gamma_5) - b_f (1+\gamma_5) \right]}_{\text{vertex}} v_{\bar\sigma}(p_{\bar f}) \cdot \varepsilon_\mu^\lambda(k)$$

Need to find:

$$\frac{1}{3} \sum_{\lambda,\sigma,\bar\sigma} |\mathcal{M}|^2 = \frac{1}{3} \sum_{\lambda,\sigma,\bar\sigma} \bar{u}_\sigma\, \gamma\cdot\varepsilon^\lambda \left[ a_f (1-\gamma_5) - b_f (1+\gamma_5) \right]$$

$\leftarrow$ sum over spins

$\uparrow$ average over initial polarizations

$$\cdot\, v_{\bar\sigma} \quad \underbrace{\bar{v}_{\bar\sigma}\gamma^0}_{v^+ \underbrace{\gamma^0\gamma^0}_{1}} \left[ a_f^* (1-\underbrace{\gamma_5^+}_{\gamma_5}) - b_f^* (1+\underbrace{\gamma_5^+}_{\gamma_5}) \right] \gamma^+ \cdot \varepsilon^{\lambda *} \gamma^0 u_\sigma$$

$\Rightarrow$ as $(\gamma^0)^+ = \gamma^0$, $(\gamma^i)^+ = -\gamma^i$, $i = 1,2,3$

$$\Rightarrow \gamma^0 \gamma_\mu^+ \gamma^0 = \begin{cases} \gamma^0 & \text{if } \mu = 0 \\ \gamma^i & \text{if } \mu = i \end{cases} \Rightarrow \gamma^0 (\gamma^\mu)^+ \gamma^0 = \gamma^\mu$$

$$\text{as } \{\gamma^\mu, \gamma^5\} = 0 \Rightarrow \gamma^0 \gamma^5 (\gamma^\mu)^+ \gamma^0 = \begin{cases} -\gamma^5\gamma^0, & \mu = 0 \\ -\gamma^5\gamma^i, & \mu = i \end{cases}$$

$$\Rightarrow \gamma^5 \to -\gamma^5$$

$$\Rightarrow \frac{1}{3}\sum_{\lambda,\sigma,\bar\sigma} |M|^2 = \frac{1}{3}\sum_{\lambda,\sigma,\bar\sigma} \bar u_\sigma(p_f)\, \gamma\cdot\varepsilon^\lambda \left[a_f(1-\gamma_5) - b_f(1+\gamma_5)\right] v_{\bar\sigma}(p_{\bar f})$$

$$\cdot\, \bar v_{\bar\sigma}(p_{\bar f}) \left[a_f^*(1+\gamma_5) - b_f^*(1-\gamma_5)\right] \gamma\cdot\varepsilon^{\lambda *}\, u_\sigma(p_f)$$

Now, use $\sum_\sigma u_\sigma(p_f)\,\bar u_\sigma(p_f) = \gamma\cdot p_f + m_f$

$$\sum_{\bar\sigma} v_{\bar\sigma}(p_{\bar f})\,\bar v_{\bar\sigma}(p_{\bar f}) = \gamma\cdot p_{\bar f} - m_f$$

$$\Rightarrow \frac{1}{3}\sum |M|^2 = \frac{1}{3}\sum_\lambda \mathrm{Tr}\Big[(\not p_f + m_f)\,\gamma\cdot\varepsilon^\lambda \left[a_f(1-\gamma_5) - b_f\cdot\right.$$

$$\left.\cdot(1+\gamma_5)\right](\not p_{\bar f} - m_f)\left[a_f^*(1+\gamma_5) - b_f^*(1-\gamma_5)\right]\gamma\cdot\varepsilon^{\lambda *}\Big] =$$

$$= \frac{1}{3}\sum_\lambda \mathrm{Tr}\Big[(\not p_f + m_f)\,\gamma\cdot\varepsilon^\lambda \Big[2\left(|a_f|^2(1-\gamma_5) + |b_f|^2(1+\gamma_5)\right)\not p_{\bar f} +$$

$$+ m_f\left(2 b_f a_f^*(1+\gamma_5) + 2 a_f b_f^*(1-\gamma_5)\right)\Big]\gamma\cdot\varepsilon^{\lambda *}\Big] =$$

$$= \left(\begin{array}{l}\text{as } m_{u,d} \ll M_Z \\ m_e \ll M_Z\end{array} \Rightarrow \text{neglect masses}\right) = \frac{2}{3}\sum_\lambda \mathrm{Tr}\Big[\not p_f\,\gamma\cdot\varepsilon^\lambda\cdot$$

$$\cdot\left(|a_f|^2(1-\gamma_5) + |b_f|^2(1+\gamma_5)\right)\not p_{\bar f}\ \gamma\cdot\varepsilon^{\lambda *}\Big].$$

In the Z-boson rest frame $\varepsilon_\mu^{(1)} = (0,1,0,0)$,

$\varepsilon_\mu^{(2)} = (0,0,1,0)$, $\varepsilon_\mu^{(3)} = (0,0,0,1)$ as $k\cdot\varepsilon = M\cdot\varepsilon_0 = 0 \Rightarrow \varepsilon_0 = 0$

$$\Rightarrow \sum_\lambda \varepsilon_\mu^\lambda \varepsilon_\nu^{\lambda *} = -g_{\mu\nu} \quad \text{for } \mu,\nu = 1,2,3 \quad (\emptyset \text{ otherwise})$$

$$\Rightarrow \frac{1}{3}\sum|M|^2 = \frac{2}{3}\,\mathrm{Tr}\left[\gamma^i\,\gamma\cdot p_f\,\gamma^i\left(|a_f|^2(1-\gamma_5) + |b_f|^2\cdot(1+\gamma_5)\right)\gamma\cdot p_{\bar f}\right]$$

massless

$$\gamma^i\,\gamma\cdot p_f\,\gamma^i = \underbrace{\gamma^i\gamma^0\gamma^i}_{3\gamma^0}\,p_f - \gamma^i\gamma^j\gamma^i\,p_f^j =$$

$$\gamma^i\gamma^j\gamma^i = \gamma^i\big(-\gamma^i\gamma^j + \underbrace{\{\gamma^j,\gamma^i\}}_{-2\delta^{ij}}\big) = +3\gamma^j - 2\gamma^j = \gamma^j$$

$$= 3\gamma^0 p_f - \vec\gamma\cdot\vec p_f$$

$$\Rightarrow \frac{1}{3}\sum|M|^2 = \frac{2}{3}\,\mathrm{Tr}\left[\left(3\gamma^0 p_f - \vec\gamma\cdot\vec p_f\right)\left(|a_f|^2(1-\gamma_5)+|b_f|^2(1+\gamma_5)\right)\cdot\gamma\cdot p_{\bar f}\right] =$$

| as $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5)=0$
$\Rightarrow$ drop $\gamma^5$'s
$\mathrm{tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$

$$= \frac{2}{3}\left(|a_f|^2+|b_f|^2\right)\cdot\left[4\cdot 3\cdot p_f\cdot p_{\bar f} - 4\cdot\vec p_f\cdot\vec p_{\bar f}\right] =$$

| Z CMS frame $\Rightarrow$
$\Rightarrow \vec p_{\bar f} = -\vec p_f$, $p_{\bar f} = p_f$

$$\Rightarrow = \frac{8}{3}\left(|a_f|^2+|b_f|^2\right)\cdot\left(3p_f^2 + \vec p_f^{\,2}\right) = \frac{32}{3}|\vec p_f|^2\left(|a_f|^2+|b_f|^2\right) = \frac{1}{3}\sum|M|^2$$

$\Rightarrow$ the decay rate:

$$d\Gamma = \frac{1}{2M_Z}\,\frac{d^3p_f}{(2\pi)^3\,2E_f}\,\frac{d^3p_{\bar f}}{(2\pi)^3\,2E_{\bar f}}\,(2\pi)^4\,\delta^{(4)}(k-p_f-p_{\bar f})\,\frac{1}{3}\sum|M|^2$$

$$\delta(M_Z - p_f - p_{\bar f})\,\delta^3(\vec p_f+\vec p_{\bar f})$$
$=$

$$\Rightarrow \Gamma = \frac{1}{2M_Z}\int\frac{d^3p_f\,d^3p_{\bar f}}{(2\pi)^6\,4\cancel{E_f}\cancel{E_{\bar f}}}\,(2\pi)^4\,\delta^{(4)}(k-p_f-p_{\bar f})\,\frac{32}{3}\cancel{|\vec p_f|^2}\cdot$$

$$\cdot\left(|a_f|^2+|b_f|^2\right) = \frac{1}{2M_Z}\,\frac{8}{3}\,\frac{1}{(2\pi)^2}\int d^3p_f \cdot \delta(M_Z-2p_f)\cdot$$

$$\cdot\left(|a_f|^2+|b_f|^2\right) = \frac{4}{3M_Z}\,\frac{1}{(\cancel{2\pi})^{\cancel{2}}}\cdot \cancel{4\pi}\cdot\left(\frac{M_Z}{2}\right)^2\cdot\frac{1}{2}\left(|a_f|^2+|b_f|^2\right)$$

$$= \frac{M_Z}{6\pi}\left[|a_f|^2+|b_f|^2\right]$$ $\Rightarrow$ finally, as $a_f, b_f$ are real $\Rightarrow$ drop $|\ldots|$ $\Rightarrow$

$$\boxed{\Gamma_{Z\to f\bar f} = \frac{M_Z}{6\pi}\left[a_f^2+b_f^2\right]}$$

a) Neutrinos: $b_\nu = 0$, $a_\nu = \dfrac{g}{4\cos\theta_W}$

$$\Rightarrow \Gamma_{Z\to\nu\bar\nu} = \frac{M_Z}{6\pi}\,\frac{g^2}{16\cos^2\theta_W} = \frac{g^2 M_Z}{96\pi\cos^2\theta_W}$$

b) Electrons: $a_e = \dfrac{g}{4\cos\theta_W}\left(2\sin^2\theta_W - 1\right)$

$$b_e = \frac{-g}{4\cos\theta_W}\,2\sin^2\theta_W$$

$$\Rightarrow \Gamma_{Z\to e^+e^-} = \frac{g^2 M_Z}{96\pi\cos^2\theta_W}\left[\left(2\sin^2\theta_W-1\right)^2 + 4\sin^4\theta_W\right]$$

$$= \frac{g^2 M_Z}{96\pi\cos^2\theta_W}\,\frac{1}{2}\left[1+\left(1-4\sin^2\theta_W\right)^2\right] = \Gamma_{Z\to\nu\bar\nu}\cdot\frac{1}{2}\left[1+\left(1-4\sin^2\theta_W\right)^2\right]$$

c) u-quarks: $a_u = \dfrac{g}{4\cos\theta_W}\left(1-\frac{4}{3}\sin^2\theta_W\right)$

$$b_u = \frac{g}{4\cos\theta_W}\,\frac{4}{3}\sin^2\theta_W$$

$$\Gamma_{Z\to u\bar{u}} = \Gamma_{Z\to\nu\bar{\nu}} \times (3\text{ colors}) \times \left[\left(1-\tfrac{4}{3}\sin^2\theta_W\right)^2 + \left(\tfrac{4}{3}\sin^2\theta_W\right)^2\right]$$

$$= \Gamma_{Z\to\nu\bar{\nu}} \cdot \frac{3}{2} \cdot \left[1 + \left(1-\tfrac{8}{3}\sin^2\theta_W\right)^2\right]$$

d) d-quarks

$$a_d = -\frac{g}{4\cos\theta_W}\left(1-\tfrac{2}{3}\sin^2\theta_W\right)$$

$$b_d = -\frac{g}{4\cos\theta_W}\,\tfrac{2}{3}\sin^2\theta_W$$

$$\Rightarrow \Gamma_{Z\to d\bar{d}} = \Gamma_{Z\to\nu\bar{\nu}} \cdot 3\left[\left(1-\tfrac{2}{3}\sin^2\theta_W\right)^2 + \left(\tfrac{2}{3}\sin^2\theta_W\right)^2\right]$$

$$= \Gamma_{Z\to\nu\bar{\nu}} \cdot \frac{3}{2} \cdot \left[1 + \left(1-\tfrac{4}{3}\sin^2\theta_W\right)^2\right].$$

The total Z-boson decay width:

$$\boxed{\Gamma_Z = \frac{g^2 M_Z}{192\pi\cos^2\theta_W}\Big\{2N_\nu + \left[1+(1-4\sin^2\theta_W)^2\right]N_e + 3\left[1+\left(1-\tfrac{8}{3}\sin^2\theta_W\right)^2\right]N_u + 3\left[1+\left(1-\tfrac{4}{3}\sin^2\theta_W\right)^2\right]N_d\Big\}}$$

as $M_Z \simeq 91$ GeV $\Rightarrow$ $N_u = 2$ (u, c only, $m_t \simeq 174$ GeV)

$N_d = 3$ $(d, s, b)$

know $N_e = 3$ $(e, \mu, \tau)$

as $\sin^2\theta_W \simeq \frac{1}{4}$ $\Rightarrow$ $\Gamma_{Z\to\nu\bar{\nu}} : \Gamma_{Z\to e^+e^-} : \Gamma_{Z\to u\bar{u}} : \Gamma_{Z\to d\bar{d}}$ =

$= 2N_\nu : N_e : \frac{10}{3}N_u : \frac{13}{3}N_d$

$\Rightarrow$ measure $\Gamma_Z^{tot} \simeq 2.5$ GeV, $\Gamma_{Z\to e^+e^-}$, $\Gamma_{Z\to u\bar{u}}$, $\Gamma_{Z\to d\bar{d}}$ $\Rightarrow$

$\Rightarrow$ since one can see $e^+e^-$ in the final state, also one can see/measure $\bar{u}u$ and $\bar{d}d$ final states (hadronic jets) $\Rightarrow$ cannot see neutrinos $\Rightarrow$ subtract visible widths from the total width to get neutrino final state width

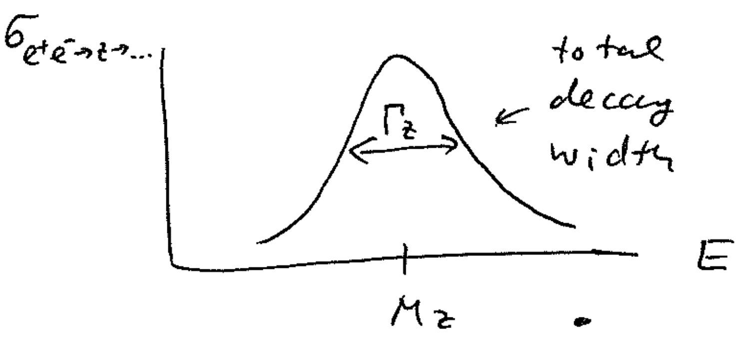
$$\Gamma_{Z\to\nu\bar{\nu}} = \Gamma_Z^{tot} - \Gamma_{Z\to e^+e^-} - \Gamma_{Z\to u\bar{u}} - \Gamma_{Z\to d\bar{d}}$$

$$\Rightarrow \quad N_\nu = \frac{\Gamma_{Z\to\nu\bar{\nu}}^{\text{experiment}}}{\Gamma_{Z\to\nu\bar{\nu}}^{\text{theory, 1 species}}}$$

where, in our calculation,

$$\Gamma_{Z\to\nu\bar{\nu}}^{\text{theory, 1 species}} = \frac{g^2 M_Z}{192\pi\cos^2\theta_W}\cdot 2 \, .$$

Linear colliders like SLAC & LEP measured $e^+e^- \to Z \to \ldots$ decays.




$\Rightarrow$ LEP result is

$$\boxed{N_\nu = 2.984 \pm 0.008}$$

consistent with 3 neutrino generations!