

Last time

Reviewed functional integral approach to quantization of field theories.

Faddeev-Popov Quantization (cont'd)

$$Z[0] = \int \mathcal{D}A_\mu e^{i \int d^4x \left(\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \right)}$$

problems: (i) integral is ∞ due to gauge fields related by gauge transformations

(ii) Quadratic term is $\frac{1}{2} A^\mu [g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu] A^\nu$

$\Rightarrow [\dots]$ has no inverse \Rightarrow no propagator!

Faddeev - Popov Quantization

(22)

ca. 1967

We want to quantize a gauge theory:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \quad (\text{consider a general non-Abelian case}).$$

The generating functional is

$$\begin{aligned} Z[0] &= \int \mathcal{D} A_\mu e^{iS} = \int \mathcal{D} A_\mu e^{i \int d^4x \left(-\frac{1}{4}\right) F_{\mu\nu}^a F^{\mu\nu a}} \\ &= \int \mathcal{D} \bar{A}_\mu e^{iS} \cdot \int \mathcal{D} \Lambda \end{aligned}$$

where \bar{A}_μ is the field in one particular gauge, Λ is the gauge transformation.

Problem: $\int \mathcal{D} \Lambda = \infty \Rightarrow Z = \infty \Rightarrow \text{bad!}$

Even worse is the need to pick a gauge: consider

$$\begin{aligned} \text{Abelian field } A_\mu: \quad \mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu - \\ &- \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{2} A^\mu \underbrace{[g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu]}_{(D^{-1})_{\mu\nu}} A^\nu \end{aligned}$$

\Rightarrow to find photon propagator need to solve

$$-i [g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu] D^{\nu\rho}(x) = \delta_\mu^\rho \delta^4(x)$$

$$\Rightarrow \text{act with } \partial^\mu \Rightarrow -i (\partial \cdot \partial^2 - \partial^2 \partial_\nu) D^{\nu\rho} = 0 = \partial^\rho \delta^4(x)$$

\Rightarrow this can not be true \Rightarrow the operator (2.3) has no inverse! \Rightarrow no photon propagator?

However, if we choose a gauge, e.g. $\partial_\mu A_\mu = 0$

$$\Rightarrow \mathcal{L} = \frac{1}{2} A^\mu \square A^\nu \Rightarrow -ig_{\mu\nu} \square D^{\nu\rho}(x) = \delta_\mu^\rho \delta^4(x).$$

\Rightarrow easy to invert!

\Rightarrow Need to fix the gauge!

Start with $Z(0) = \int \mathcal{D}A_\mu e^{iS}$

Insert into the integrand $(A_\mu^\Lambda = \Lambda A_\mu + \frac{i}{g}(\partial_\mu \Lambda) \wedge)$

$$1 = \int \mathcal{D}\Lambda \delta(G(\Lambda)) = \int \mathcal{D}\Lambda \cdot \delta(G(A^\Lambda)) \det \left(\frac{\delta G(A^\Lambda)}{\delta \Lambda} \right)$$

where $G(A) = 0$ is the gauge condition we want to impose, e.g. $G(A) = \partial_\mu A^\mu$ for covariant gauge. Now

$$Z(0) = \int \mathcal{D}A_\mu e^{iS(A_\mu)} \left(\int \mathcal{D}\Lambda \delta(G(A^\Lambda)) \det \left(\frac{\delta G(A^\Lambda)}{\delta \Lambda} \right) \right)_{\Lambda=1}$$

Change the order of integration & define a new

field $A'_\mu = A_\mu^\Lambda$ to write (dropping the prime)

(as in QED $\mathcal{D}A_\mu = \mathcal{D}A'_\mu$, $S(A_\mu) = S(A'_\mu)$) $\Lambda = \text{unitary}$

$$Z(\eta) = \int \mathcal{D}\Lambda \cdot \int \mathcal{D}A_\mu e^{iS(A_\mu)} \delta(G(A_\mu)) \det\left(\frac{\delta G(A_\mu)}{\delta \Lambda}\right) \quad (24)$$

still ∞ , but
 an overall factor \Rightarrow cancels in correlators like

we assumed that this is independent of Λ , only after $A_\mu \rightarrow A$

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{1}{Z} \cdot \int \mathcal{D}A_\mu A_\mu(x) A_\nu(y) e^{iS}$$

A trick: choose $G(A) = \partial_\mu A^\mu - \omega^a(x) \Rightarrow$

$$\Rightarrow \delta(G(A)) = \delta(\overbrace{\partial_\mu A^\mu}^{\bar{G}(A)} - \omega^a(x))$$

Nothing else in Z depends on $\omega^a(x) \Rightarrow$ integrate

over $\omega(x)$: $1 = \underbrace{N(\zeta)}_{\text{norm}} \int \mathcal{D}\omega^a(x) e^{-i \int d^4x \frac{\omega^2}{2\zeta}}$

$$\Rightarrow Z(\eta) = \int \mathcal{D}\Lambda \cdot N(\zeta) \int \mathcal{D}A_\mu e^{iS(A_\mu)} \int \mathcal{D}\omega^a e^{-i \int d^4x \frac{\omega^2}{2\zeta}}$$

$$\delta(\bar{G}(A) - \omega(x)) \det\left(\frac{\delta G(A_\mu)}{\delta \Lambda}\right) = \int \mathcal{D}\Lambda \cdot N(\zeta) \cdot$$

$$\int \mathcal{D}A_\mu \cdot \det\left(\frac{\delta G(A_\mu)}{\delta \Lambda}\right) \cdot e^{iS(A_\mu)} \cdot e^{-i \int d^4x \frac{1}{2\zeta} (\bar{G}(A))^2}$$

drop $\Lambda = \Lambda$ as δ -fn insures it

$\int \mathcal{D}\Lambda \cdot N(\zeta)$ is an unimportant overall factor.

What do we do with $\det\left(\frac{\delta G}{\delta \Lambda}\right)$?

We have

(25)

$$Z(\theta) \sim \int \mathcal{D}A_\mu \det\left(\frac{\delta G(A^\mu)}{\delta A}\right) \cdot e^{i S(A) - i \int d^4x \frac{[G(A)]^2}{2\zeta}}$$

We want to put det into the exponent ~ make it a part of the Lagrangian.

Note that $\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}} \Rightarrow$

$$\Rightarrow \int_{-\infty}^{\infty} dx_1 \dots dx_n e^{-a_1 x_1^2 - \dots - a_n x_n^2} = \left(\frac{\pi}{a}\right)^{n/2} \frac{1}{\sqrt{a_1 a_2 \dots a_n}}$$

$$= \frac{\pi^{n/2}}{\sqrt{\det A}}, \quad A = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \dots & a_n \end{pmatrix} \text{ a diagonal matrix.}$$

Similarly, for $\forall A$ get

$$\int_{-\infty}^{\infty} d^4x e^{-x^T A x} = \frac{\pi^{n/2}}{\sqrt{\det A}}$$

\Rightarrow can absorb $\frac{1}{\sqrt{\det A}}$ into exponent. But here

we have

$\det A!$

$$\left\{ \begin{array}{l} \text{Grassmann #'s: } \eta \cdot \theta = -\theta \cdot \eta \text{ (ant: -commute)} \\ \Rightarrow \theta^2 = 0, \eta^2 = 0 \end{array} \right.$$

Grassmann quantities: η is a Grassmann #

$A, B = \text{regular or Grassmann #'s}$

\Rightarrow if η is single-component $\Rightarrow f(\eta) = A + B\eta$

$$(\eta^2 = 0, \eta^3 = 0, \dots) \Rightarrow \frac{df}{d\eta} = B \Rightarrow \frac{d^2 f}{d\eta^2} = 0$$

\uparrow if $B = \text{complex (regular) \#}$

\Rightarrow no inverse to differentiation? (20)

To define integrals note:

$$\int d\eta f(\eta) = \int d\eta (A + B\eta) = \int \eta \rightarrow \eta + \theta = \int d\eta (A + B\eta + B\theta)$$

$$\Rightarrow \int d\eta \cdot A = \int d\eta (A + B\theta) \Rightarrow \int d\eta = 0$$

$$\int d\eta B\eta = B \quad (\text{linear in } B, \text{ adjust constant to } 1)$$

$$\Rightarrow \int d\eta \cdot \eta = 1 \quad \text{Note that } \int d\theta \int d\eta \cdot \eta \cdot \theta = +1.$$

Complex $\eta = \frac{\eta_1 + i\eta_2}{\sqrt{2}}$ is c.c. $\bar{\eta} = \frac{\eta_1 - i\eta_2}{\sqrt{2}}$, $\eta_1, \eta_2 =$ Grassmann #'s, $\eta^2 = 0, \bar{\eta}^2 = 0, \bar{\eta}\eta \neq 0$
 $\int d\eta = \int d\bar{\eta} = 0, \int d\eta \cdot \eta = \int d\bar{\eta} \bar{\eta} = 1.$

$$\int d\bar{\eta} d\eta e^{-b\bar{\eta}\eta} = \int d\bar{\eta} \int d\eta (1 - b\bar{\eta}\eta) = \int d\bar{\eta} \cdot (+)b\bar{\eta} = 1b = b.$$

Two-component Grassmann #'s: $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \bar{\eta} = \begin{pmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \end{pmatrix}$

$$\Rightarrow \bar{\eta}^T \eta = \bar{\eta}_1 \eta_1 + \bar{\eta}_2 \eta_2$$

$$(\bar{\eta}^T \eta)^2 = (\bar{\eta}_1 \eta_1 + \bar{\eta}_2 \eta_2)^2 = 2\bar{\eta}_1 \eta_1 \bar{\eta}_2 \eta_2$$

$\int d\bar{\eta} d\eta e^{-\bar{\eta}^T M \eta}$ ~ need to find, M is a 2×2 matrix

$$\Rightarrow e^{-\bar{\eta}^T M \eta} = e^{-\begin{pmatrix} \bar{\eta}_1 & \bar{\eta}_2 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}}$$

$$= \exp \left\{ -\begin{pmatrix} \bar{\eta}_1 & \bar{\eta}_2 \end{pmatrix} \begin{pmatrix} M_{11} \eta_1 + M_{12} \eta_2 \\ M_{21} \eta_1 + M_{22} \eta_2 \end{pmatrix} \right\} = \exp \left\{ -\left[\bar{\eta}_1 (M_{11} \eta_1 + M_{12} \eta_2) + \bar{\eta}_2 (M_{21} \eta_1 + M_{22} \eta_2) \right] \right\}$$

$$+ \bar{\eta}_2 (M_{21} \eta_1 + M_{22} \eta_2) \Big] \Big\} = 1 - \bar{\eta}_1 (M_{11} \eta_1 + M_{12} \eta_2) \quad (27)$$

$$- \bar{\eta}_2 (M_{21} \eta_1 + M_{22} \eta_2) + \frac{1}{2} 2 \bar{\eta}_1 (M_{11} \eta_1 + M_{12} \eta_2) \bar{\eta}_2 (M_{21} \eta_1 + M_{22} \eta_2)$$

$$+ M_{22} \eta_2) = 1 - \bar{\eta}_1 (M_{11} \eta_1 + M_{12} \eta_2) - \bar{\eta}_2 (M_{21} \eta_1 + M_{22} \eta_2)$$

$$+ \bar{\eta}_1 \eta_1 \bar{\eta}_2 \eta_2 (M_{11} M_{22} - M_{12} M_{21})$$

$$\Rightarrow \int d\bar{\eta} d\eta e^{-\bar{\eta}^T M \eta} = \int \overbrace{d\bar{\eta}_1 d\eta_1 d\bar{\eta}_2 d\eta_2}^{d\bar{\eta}_1 d\eta_1 d\bar{\eta}_2 d\eta_2} \bar{\eta}_1 \eta_1 \bar{\eta}_2 \eta_2 \cdot$$

$$\cdot (M_{11} M_{22} - M_{12} M_{21}) = \det M.$$

\Rightarrow can show for \forall dimension

$$\int d\bar{\eta} d\eta e^{-\bar{\eta}^T M \eta} = \det M$$

$$\Rightarrow \det \left[\frac{\delta G(A^a)}{\delta \eta^a} \right] = \int d\bar{\eta} \mathcal{D} \bar{\eta} e^{-i \int d^4x \bar{\eta}^a \frac{\delta G}{\delta \eta^a} \eta^a}$$

$$\Rightarrow Z(0) \sim \int \mathcal{D} A_\mu \overbrace{\mathcal{D} \bar{\eta} \mathcal{D} \eta}^{i \mathcal{G}(A)} e^{-i \int d^4x \frac{[\mathcal{G}(A)]^2}{2\xi}} = i \int d^4x \bar{\eta} \frac{\delta G}{\delta \eta} \eta$$

η^a Faddeev-Popov ghost, has a color index $a = 1, \dots, N^2 - 1$ for $SU(N)$

Covariant gauge: $\mathcal{G}(A) = \partial_\mu A^{\mu a} \Rightarrow$ gauge

transform is $A_\mu \rightarrow \Lambda A_\mu \Lambda^{-1} - \frac{i}{g} (\partial_\mu \Lambda) \Lambda^{-1}$

\Rightarrow for infinitesimal gauge transform: (28)

(need $\frac{\delta G(A^\mu)}{\delta \Lambda} \Rightarrow$ vary $\Lambda \rightarrow \Lambda + \delta \Lambda$, below A_μ is A_μ^Λ)

$$\Lambda = 1 + i \alpha^a t^a \Rightarrow A_\mu^a t^a \rightarrow (1 + i \alpha^a t^a) A_\mu$$

$$(1 - i \alpha^b t^b) - \frac{i}{g} i t^a (\partial_\mu \alpha^a) (1 - i \alpha^b t^b) =$$

$$= A_\mu + i [\alpha, A_\mu] + \frac{1}{g} \partial_\mu \alpha = t^a A_\mu^{i a}$$

$$\Rightarrow A_\mu^{i a} = A_\mu^a + i \cdot i f^{abc} \alpha^b A_\mu^c + \frac{1}{g} \partial_\mu \alpha^a$$

$$= A_\mu^a + f^{abc} A_\mu^b \alpha^c + \frac{1}{g} \partial_\mu \alpha^a =$$

$$= A_\mu^a + \frac{1}{g} D_\mu \alpha^a, \text{ where } D_\mu \alpha^a = \partial_\mu \alpha^a + g f^{abc} A_\mu^b \alpha^c$$

$$\Rightarrow \frac{\delta G}{\delta \Lambda} = \frac{\delta G}{\delta \alpha} = \frac{\delta (\partial_\mu A^{a\mu})}{\delta \alpha} = \partial_\mu \frac{1}{g} D^\mu$$

\leftarrow absorb by re-defining ξ

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^{a\mu})^2 - \bar{\psi} \partial_\mu \psi^{\mu}$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^{a\mu})^2 + (\partial_\mu \bar{\psi}^a) (D^\mu \psi)^a$$

Faddeev-Popov Lagrangian

Light-cone gauge: $\bar{G}(A) = n \cdot A^a = n_\mu A^{a\mu} \Rightarrow$

$$\Rightarrow \frac{\delta G}{\delta \alpha} = n_\mu \frac{\delta A^{a\mu}}{\delta \alpha} = n_\mu \frac{1}{g} D^\mu = \frac{1}{g} n_\mu (\partial^\mu - ig [A^\mu, \dots])$$

(in the $\xi \rightarrow 0$ limit) \longrightarrow as $n \cdot A = 0$

$= \frac{1}{g} n \cdot \partial \Rightarrow$ is $A_\mu \sim$ independent \Rightarrow (24)

\Rightarrow ~~gives~~ gives only an overall factor in \int

\Rightarrow do not need it, no need to introduce the ghost!

\Rightarrow no ghosts in light-cone gauge!

$\Rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi} (n \cdot A)^2$ LC gauge Lagrangian.

($\xi \rightarrow 0$ is implicit)

\Rightarrow unlike QED case, in QCD in covariant gauge $\frac{\delta G(A^a)}{\delta A}$ depends on A_μ and can not be

taken out of DA_μ integral \Rightarrow have to introduce the ghost field!