Last time | Faddeev - Popov Quantization (cont'd)

$$Z[0] = \int \mathcal{D} A_\mu \, e^{iS} \quad \sim \text{hard to work with}$$

We fixed the gauge condition $G(A_\mu) = 0$ and obtained

$$Z[0] = \int \mathcal{D}\Lambda \, N(\xi) \int \mathcal{D} A_\mu \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) e^{iS[A_\mu] - i\int d^4x \frac{1}{2\xi}(\partial_\mu A^{a}_\mu)^2}$$

for $\partial_\mu A^{a\mu} - \omega^a = G(A_\mu) = 0$ gauge condition.

Further, to put the determinant into the exponent, we introduced a Grassmann variable ghost field $\eta^a(x)$, $a = 1, \ldots, N^2 - 1$ for $SU(N)$, and got

$$Z[j^a_\mu] = \int \mathcal{D} A_\mu \mathcal{D}\eta \mathcal{D}\bar{\eta} \, e^{i\int d^4x [\mathcal{L}_{FP} + j^a_\mu A^{a\mu}]}$$

where the Faddeev-Popov Lagrangian is

$$\mathcal{L}_{FP} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + \partial_\mu \bar{\eta}^a (D^\mu \eta)^a.$$

$$D_\mu \alpha^a \equiv \partial_\mu \alpha^a + g f^{abc} A^b_\mu \alpha^c, \quad D_\mu \alpha = \partial_\mu \alpha - ig[A_\mu, \alpha]$$

$\Rightarrow$ for infinitesimal gauge transform:

(need $\frac{\delta G(A^\Lambda)}{\delta \Lambda}$ $\Rightarrow$ vary $\Lambda \to \Lambda + \delta\Lambda$, below A_μ is A_μ^Λ)

$$\Lambda = 1 + i\,\alpha^a t^a \quad \Rightarrow \quad A_\mu^a t^a \to (1 + i\,\alpha^a t^a) A_\mu \cdot$$

$$\cdot (1 - i\,\alpha^b t^b) - \frac{i}{g}\, i\, t^a (\partial_\mu \alpha^a)(1 - i\,\alpha^b t^b) =$$

$$= A_\mu + i[\alpha, A_\mu] + \frac{1}{g}\partial_\mu \alpha = t^a A_\mu'^a$$

$$\Rightarrow \quad A_\mu'^a = A_\mu^a + i \cdot i f^{abc} \alpha^b A_\mu^c + \frac{1}{g}\partial_\mu \alpha^a$$

$$= A_\mu^a + f^{abc} A_\mu^b \alpha^c + \frac{1}{g}\partial_\mu \alpha^a =$$

$$= A_\mu^a + \frac{1}{g} D_\mu \alpha^a, \text{ where } D_\mu \alpha^a = \partial_\mu \alpha^a + g f^{abc} A_\mu^b \alpha^c$$

$$\Rightarrow \quad \frac{\delta G}{\delta \Lambda} = \frac{\delta G}{\delta \alpha} = \frac{\delta(\partial_\mu A^{a\mu})}{\delta \alpha} = \partial_\mu \frac{1}{g} D^\mu$$

$\uparrow$ absorb by re-defining η

$$\Rightarrow \quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 - \bar{\eta}\, \partial_\mu D^\mu \eta$$

$$\Rightarrow \quad \boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + (\partial_\mu \bar{\eta}^a)(D^\mu \eta)^a}$$

Faddeev - Popov Lagrangian

<u>Light-cone gauge:</u> $\bar{G}(A) = n \cdot A^a = n_\mu A^{a\mu} \quad \Rightarrow$

$$\Rightarrow \quad \frac{\delta G}{\delta \alpha} = n_\mu \frac{\delta A^{a\mu}}{\delta \alpha} = n_\mu \frac{1}{g} D^\mu = \frac{1}{g} n_\mu (\partial^\mu - ig[A^\mu, \ldots])$$

$A^\mu \to 0$

(in the $\xi \to 0$ limit) $\longrightarrow$ as $n \cdot A = 0$

$= \frac{1}{g} n \cdot \partial$ ⇒ is A_μ - independent ⇒

⇒ ~~again~~ gives only an overall factor in

⇒ do not need it, no need to introduce the ghost!

⇒ no ghosts in light-cone gauge!

$$\Rightarrow \quad \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (n \cdot A^a)^2$$

LC gauge Lagrangian.

($\xi \to 0$ is implicit)

⇒ unlike QED case, in QCD in covariant gauge $\frac{\delta G(A^\Lambda)}{\delta \Lambda}$ depends on A_μ and can not be taken out of DA_μ integral ⇒ have to introduce the ghost field!

$$\delta(\eta \cdot A) = \lim_{\xi \to 0} e^{-\frac{i}{2\xi} \int d^4x \, (\eta \cdot A)^2} \cdot \tilde{N}(\xi)$$

norm

~ here we use this to exponentiate the gauge-fixing δ-fctn.

Feynman Rules in QCD

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^f [i\gamma^\mu D_\mu - m_f] q^f - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}}$$

However, this Lagrangian is gauge-invariant

$$\begin{cases} A_\mu \to S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S) S^{-1} \\ q \to S q \end{cases}$$

$\Rightarrow$ need to fix the gauge!

(i) Covariant (Lorenz) gauge $\partial_\mu A^{a\mu} = 0$

$\Rightarrow$ to fix the gauge need to introduce the so-called ghost fields:

$$\mathcal{L}^{cov.gauge}_{QCD} = \bar{q}^f [i\gamma^\mu D_\mu - m_f] q^f - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})(\partial_\nu A^{a\nu}) + \partial_\mu \bar{\eta}^a (\mathcal{D}^\mu \eta)^a$$

η^a is a scalar field ~ Faddeev-Popov ghost

η^a is an anti-commuting field (Grassmann variables) (quantized like a fermion) $\Rightarrow$ unphysical $\Rightarrow$ ghosts

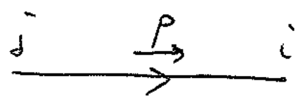
$\bar{\eta}^a$ is c.c. of η^a ; $\boxed{D_\mu = \partial_\mu - i g A_\mu}$

$$\eta = \sum_{a=1}^{8} t^a \eta^a \,, \qquad D_\mu \eta = \partial_\mu \eta - ig \underbrace{[A_\mu, \eta]}$$

note the commutator!

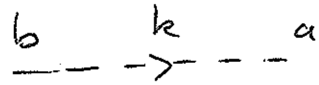
$$D_\mu \eta^a = \partial_\mu \eta^a + g f^{abc} A_\mu^b \eta^c$$

Feynman Rules:

Quark Propagator: 

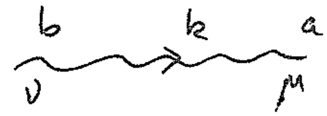
$$\frac{i}{\gamma\cdot p - m}\,\delta_{ij}$$

$$= \frac{i(\gamma\cdot p + m)}{p^2 - m^2 + i\varepsilon}\,\delta_{ij}$$

Ghost Propagator: 

$$\frac{i}{k^2 + i\varepsilon}\,\delta_{ab}$$

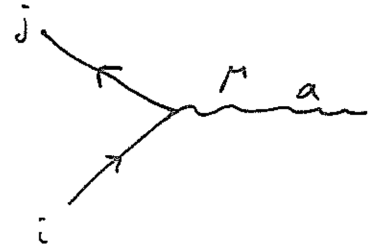
$\bar{\eta}\eta$

Gluon Propagator: 

$$\frac{-i}{k^2 + i\varepsilon}\,\delta^{ab}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

$\xi = 0$ Landau gauge

$\xi = 1$ Feynman gauge

Quark-Gluon Vertex: 

$$ig\,\gamma^\mu\,(t^a)_{ji}$$

Other interaction vertices are less trivial:

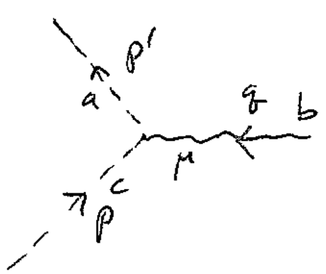
$$\partial_\mu \bar\eta\, D^\mu \eta = \partial_\mu \bar\eta\, \partial^\mu \eta \underbrace{-ig\, \partial_\mu \bar\eta\, [A^\mu, \eta]}_{\text{ghost-gluon interaction}}$$

$$\Rightarrow \mathcal{L}_{gG} = -ig\, \partial_\mu \bar\eta\, [A^\mu, \eta] = \cancel{-i}g\, \partial_\mu \bar\eta^a\, \cancel{i} f^{abc} A^{b\mu} \eta^c$$

$$= g \left(\partial_\mu \bar\eta^a\right) f^{abc} A^{b\mu} \eta^c$$

When contracting with $\eta(y)$: $\overbrace{\eta(y)\; g f^{abc} \left(\partial_\mu \bar\eta^a_{(x)}\right)}^{= e^{-ip'\cdot(y-x)}} A^{b\mu} \eta^c$, with $\partial_\mu \to ip'_\mu$

$$\Rightarrow i g f^{abc} p'_\mu \otimes \overset{\text{from } iS}{i} \Rightarrow -g f^{abc} p'_\mu = g f^{bac} p'_\mu$$



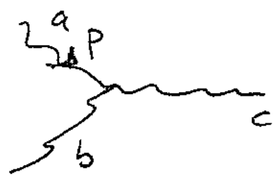
is the contribution of ghost - gluon vertex

3 - gluon vertex: $-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) \cdot g \cdot$

$$f^{abc} A^b_\mu A^c_\nu = -g\, \partial_\mu A^a_\nu\, f^{abc} A^b_\mu A^c_\nu \qquad \sim e^{-ip\cdot(x-y)}$$

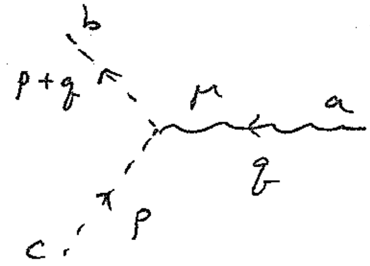
$\Rightarrow$ contracting with $A^d_\rho(y)$: $-g f^{abc} A^b_\mu A^c_\nu\, \partial_\mu \overbrace{A^a_\nu(x) A^d_\rho(y)}$, with $\partial_\mu \to -ip_\mu$

$$\Rightarrow \text{get terms like } \overset{\text{from } iS}{i}\, i g f^{abc} p_\mu = g f^{bac} p_\mu + \ldots$$



$\Rightarrow$ let us summarize all this:

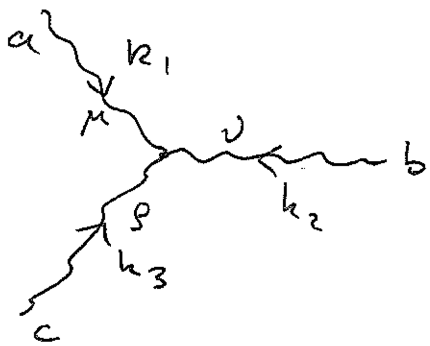
Ghost-gluon Vertex:



$$g\,(p+q)_\mu\, f^{abc}$$

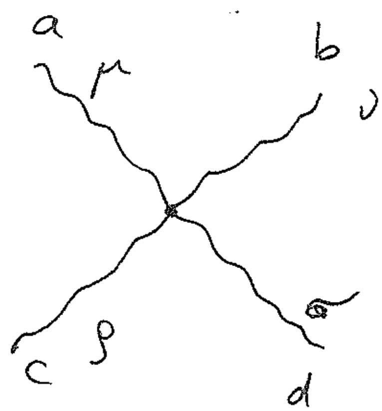
(counter-clockwise)

3-Gluon Vertex:



$$-g\, f^{abc}\left[(k_1-k_3)_\nu\, g_{\mu\rho} + (k_2-k_1)_\rho\, g_{\mu\nu} + (k_3-k_2)_\mu\, g_{\nu\rho}\right]$$

4-Gluon Vertex:



$$-ig^2\Big[f^{abe} f^{cde}\cdot(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) + f^{ace} f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}) + f^{ade} f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\Big].$$

~ same as QED for external fermions, bosons (no external ghosts), internal integrals, "−" for each fermion (or ghost) loop.

(ii) Light-cone gauge

Define light-cone variables: $A^{\pm} = \frac{A^0 \pm A^3}{\sqrt{2}}$

(choose a "preferred direction" ~ x3)

$A^+ = 0$ gauge is called the light-cone (LC) gauge

Write the gauge condition as

$\eta \cdot A = 0$ with $\eta^- = 1$, $\eta^+ = 0$, $\eta^1 = \eta^2 = 0$

$A_\mu B^\mu = A^+ B^- + A^- B^+ - A^1 B^1 - A^2 B^2$ (check)

$$\eta \cdot A = \underset{0}{\underset{\parallel}{\eta^+}} A^- + \underset{1}{\underset{\parallel}{\eta^-}} A^+ - \underset{0}{\underset{\parallel}{\eta^1}} A^1 - \underset{0}{\underset{\parallel}{\eta^2}} A^2 = A^+$$

=> there is no ghost in LC gauge!

Feynman rules: the same, but no ghost

=> no ghost propagator, no ghost-gluon vertex

=> gluon propagator is different:

$$\underset{\mu}{\overset{a}{\sim\!\sim}}\overset{k}{\to}\underset{\nu}{\overset{b}{\sim\!\sim}} \qquad \frac{-i}{k^2 + i\varepsilon}\,\delta^{ab}\left[g_{\mu\nu} - \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{\eta \cdot k}\right].$$

Some properties of light-cone coordinates $A^\pm$:

Boost along the z axis:

$$\begin{pmatrix} A'^0 \\ A'^1 \\ A'^2 \\ A'^3 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

$$\Rightarrow \quad A'^+ = \frac{A'^0 + A'^3}{\sqrt{2}} = \frac{\gamma A^0 + \beta\gamma A^3 + \beta\gamma A^0 + \gamma A^3}{\sqrt{2}} =$$

$$= \gamma(1+\beta) A^+$$

$$\Rightarrow \begin{cases} A'^+ = \gamma(1+\beta) A^+ \\ A'^- = \gamma(1-\beta) A^- \\ A'^1 = A^1 \\ A'^2 = A^2 \end{cases} \quad \Rightarrow \quad A^+A^- \to \gamma^2(1+\beta)(1-\beta) A^+A^- = A^+A^- \Rightarrow \text{invariant}$$

$\Rightarrow$ any A^+B^- is invariant under boosts in z-direction

also $\frac{A^+}{B^+}$ is invariant under —"—