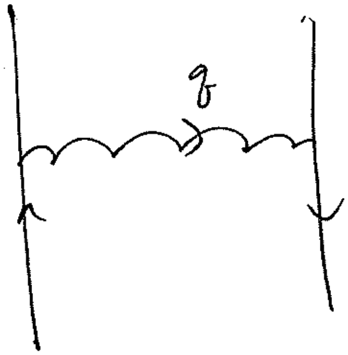
Last time | Heavy Quark Potential



$\mathcal{M} \sim -\tilde{V}(q)$

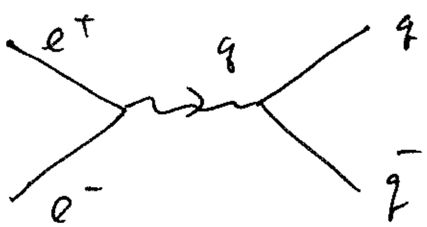
got $\tilde{V}(q) = -\dfrac{g^2 C_F}{\vec{q}^{\,2}}$

such that

$$V(r) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}}\, \tilde{V}(q) = -\frac{\alpha_s C_F}{r}$$

at short distances the potential is Coulomb-like

$e^+e^- \to$ hadrons



$Q^2 = q^2$

$$\Rightarrow \quad R(Q^2) = \frac{\sigma_{e^+e^- \to \text{hadrons}}}{\sigma_{e^+e^- \to \mu^+\mu^-}} = 3 \sum_f e_f^2$$

$\sim$ $R(Q^2)$ gets a step-like structure as Q^2 crosses various $4m_f^2$ heavy quark pair production thresholds.

Running Coupling and Asymptotic Freedom

g ~ is the coupling constant

put $m_f = 0$ in $\mathcal{L}_{QCD}$ for simplicity:

$$\mathcal{L}_{QCD}^{m_f=0} = \bar{q}^f \, i\gamma^\mu D_\mu q^f - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$$

g is the only parameter for such theory.

$\Rightarrow$ When people do perturbation theory, infinities arise:  $\sim \int^\mu \frac{d^4k}{k^4} \sim \ln\mu$ with μ a UV cutoff

- problems are usually in the ultraviolet (UV) where momenta are large
- one has to introduce a UV cutoff μ $\Rightarrow$

$\Rightarrow$ $\mathcal{L}$ & observables would depend on μ:

$$\mathcal{L} = \mathcal{L}(g, \mu), \quad \underset{\substack{\uparrow \\ \text{observable}}}{\mathcal{M}} = \mathcal{M}(g, \mu).$$

$\Rightarrow$ but physics should not be dependent on any cutoff if the theory is consistent $\Rightarrow$

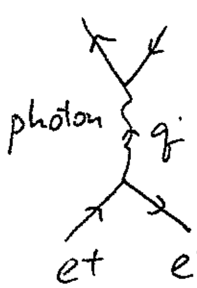
$\Rightarrow$ the only way to make it work is to have g depend on μ $\Rightarrow$ $\mathcal{L} = \mathcal{L}(g_\mu, \mu)$

$\mathcal{M} = \mathcal{M}(g_\mu, \mu)$. ← re-arrange the expansion in pert. theory to expand in g_μ.

$\Rightarrow$ <u>running coupling</u>: g_μ depends on momentum scale μ.

$\Rightarrow$ imagine an observable $\mathcal{M}$ which depends on a single four-momentum squared: $Q^2 = q_\mu q^\mu$

<u>Example</u>: $e^+e^- \to$ hadrons



$\Rightarrow$ the cross section depends on center of mass energy $Q^2 = q_\mu q^\mu \Rightarrow \sigma = \sigma(Q^2)$

in CM frame $q^\mu = (Q, \vec{0}) \Rightarrow q^2 = Q^2$.

$Q^2\sigma$ is dimensionless; quark masses $= 0$ ← simplicity
electron mass $= 0$.

$\Rightarrow$ in general would have $M = M(Q^2, \alpha_\mu, \mu)$

where $\alpha_\mu = \frac{g_\mu^2}{4\pi}$

$\Rightarrow$ assume that M is dimensionless $\Rightarrow M = M(\frac{Q^2}{\mu^2}, \alpha_\mu)$.

But: no physical observable should depend on μ!

$$\Rightarrow \boxed{\mu^2 \frac{d}{d\mu^2} M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0}$$

$$\Rightarrow \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{d\alpha_\mu}{d\mu^2} \frac{\partial}{\partial \alpha_\mu}\right] M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0$$

Def. Beta-function of QCD: $\beta(\alpha_\mu) = \mu^2 \frac{d\alpha_\mu}{d\mu^2}$.

$\beta(\alpha_\mu)$ is dimensionless $\Rightarrow$ can not depend on μ explicitly, μ-dependence comes in through α_μ only!

$$\Rightarrow \boxed{\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu}\right] M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0}$$

renormalization group equation (Callan, Symanzik '70)

~ tells how things change with the changing momentum scale / distance resolution

$\Rightarrow$ equivalently $\left[-Q^2 \frac{\partial}{\partial Q^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu}\right] M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0$.

To solve the renormalization group (RG) equation define $\rho(\alpha_\mu) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')}$

α_0 ← arbitrary cutoff

Def. Running Coupling by:

$$\boxed{\alpha(Q^2) \equiv \rho^{-1}\left(\ln \frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right)}$$

ρ^{-1} ~ inverse function

⇒ note that

(i) $\alpha(\mu^2) = \alpha_\mu$

(ii) $\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu}\right] \alpha(Q^2) = 0$

(ii) is true because $\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu}\right]$

$$\cdot\left(\ln \frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right) = -1 + \beta(\alpha_\mu) \underbrace{\frac{\partial \rho(\alpha_\mu)}{\partial \alpha_\mu}}_{1/\beta(\alpha_\mu) \text{ by definition}} = 0$$

as $M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right)$ does not depend on μ we can put $\mu = Q$ and get:

$$M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = M\left(\frac{Q^2}{\mu^2}, \alpha(\mu^2)\right) \overset{\mu^2 \to Q^2}{=} M(1, \alpha(Q^2)) = M(\alpha(Q^2))$$

⇒ any M which is a function of $\alpha(Q^2)$ only automatically satisfies RG equation.

$\Rightarrow$ We have shown that running coupling $\alpha(Q^2)$ satisfies RG equation + allows any observable dependent on it to satisfy RG equation.

$\Rightarrow$ let's find $\alpha(Q^2)$: to do this need $\rho(\alpha_\mu)$.

To find $\rho(\alpha_\mu)$ need $\beta(\alpha_\mu)$ ~ the beta-function.

Beta-function has to be found through an explicit (hard) calculation ~ see field theory texts like Peskin.

$\Rightarrow$ in perturbation theory one usually gets:

$$\boxed{\beta(\alpha) = -\beta_2\,\alpha^2 - \beta_3\,\alpha^3 + \ldots}$$

(perturbative / small coupling α expansion)

in QCD $\quad \beta_2 = \dfrac{11\,N_c - 2\,N_f}{12\pi}$, N_c ~ # colors

N_f ~ # flavors

(Politzer '73, Gross & Wilczek '73) $\Leftarrow$ Nobel Prize 2004

~ was probably obtained before by 't Hooft (oral communication)

$\Rightarrow$ it is <u>very important</u> that in QCD

$\beta(\alpha) < 0$ ~ beta-function is negative c.f. in QED have $\beta_2^{QED} = -\frac{1}{3\pi}$ such that

$\beta_2^{QED}(\alpha) > 0$.

$\Rightarrow$ why does this matter? Let's do the calculation at small coupling: put $\beta(\alpha) = -\beta_2 \alpha^2$

$$\Rightarrow \rho(\alpha_\mu) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} = -\frac{1}{\beta_2} \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\alpha'^2} = -\frac{1}{\beta_2}\left(-\frac{1}{\alpha'}\right)\Bigg|_{\alpha_0}^{\alpha_\mu} =$$

$$= \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right).$$

The inverse function: $\rho(\alpha) = w \Rightarrow \alpha = \rho^{-1}(w)$

$$\Rightarrow \frac{1}{\beta_2}\left(\frac{1}{\alpha} - \frac{1}{\alpha_0}\right) = w \Rightarrow \frac{1}{\alpha} = \frac{1}{\alpha_0} + \beta_2 w \Rightarrow$$

$$\Rightarrow \alpha = \rho^{-1}(w) = \frac{1}{\frac{1}{\alpha_0} + \beta_2 w}$$

$$\Rightarrow \alpha(Q^2) = \rho^{-1}\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right) = \frac{1}{\frac{1}{\alpha_0} + \beta_2\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right)}$$

α_0 cancels - not important

$$= \frac{1}{\cancel{\frac{1}{\alpha_0}} + \beta_2\left(\ln\frac{Q^2}{\mu^2} + \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \cancel{\frac{1}{\alpha_0}}\right)\right)} = \frac{1}{\frac{1}{\alpha_\mu} + \beta_2 \ln\frac{Q^2}{\mu^2}}$$

$$\Rightarrow \boxed{\alpha(Q^2) = \frac{\alpha_\mu}{1 + \alpha_\mu \beta_2 \ln\frac{Q^2}{\mu^2}}}$$

1-loop running coupling in a gauge theory.

$\Rightarrow$ at large distances / small Q^2 the coupling gets large $\Rightarrow$ pert. th'y breaks down, no one knows what $\alpha_s(Q^2)$ is there.

$\Rightarrow$ when does this happen? write

$$\alpha_s(Q^2) = \frac{\alpha_\mu}{1+\alpha_\mu \beta_2 \ln \frac{Q^2}{\mu^2}} = \frac{1}{\beta_2 \ln \frac{Q^2}{\Lambda^2} + \underbrace{\frac{1}{\alpha_\mu} - \beta_2 \ln \frac{\mu^2}{\Lambda^2}}_{=\,0}}$$

define the scale Λ by requiring

$$\Rightarrow \frac{1}{\alpha_\mu} = \beta_2 \ln \frac{\mu^2}{\Lambda^2} \Rightarrow \Lambda^2 = \mu^2 e^{-\frac{1}{\beta_2 \alpha_\mu}} \Rightarrow$$

$\Rightarrow \Lambda^2$ is μ-independent (check).

$$\boxed{\alpha_s(Q^2) = \frac{1}{\beta_2 \ln \frac{Q^2}{\Lambda^2}}}$$

$\Rightarrow$ coupling gets large at $Q^2 \simeq \Lambda^2$.

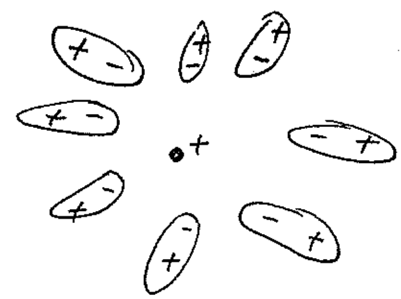
$\Rightarrow \Lambda^2$ is the fundamental parameter in QCD, usually denoted Λ^2_{QCD}.

$\Lambda_{QCD} \simeq 200$ MeV (depends on scale)

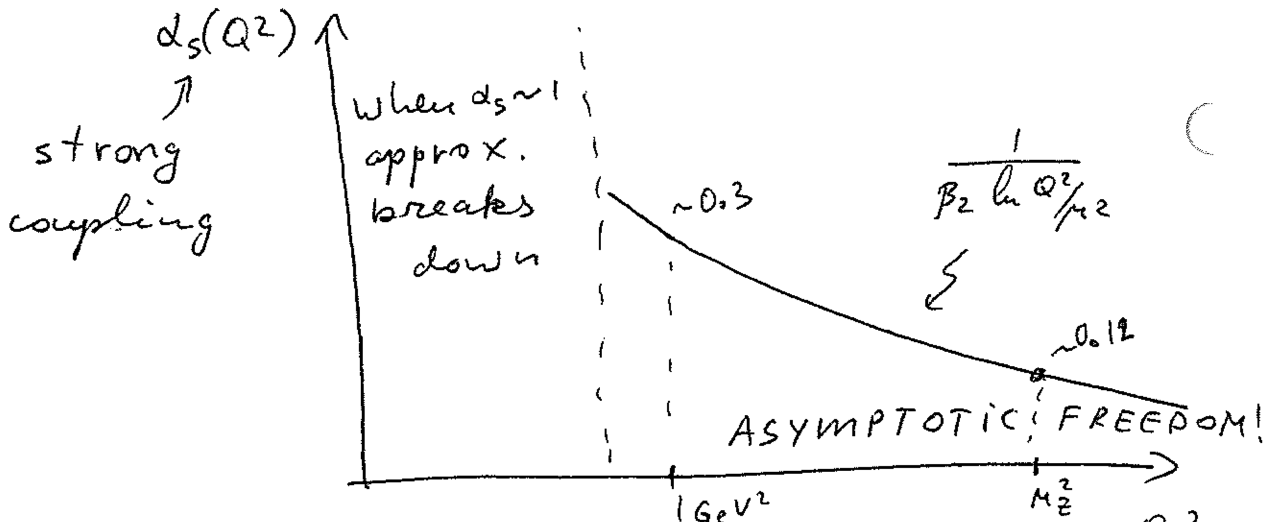
(Landau pole: $\alpha_s(\Lambda^2) = \infty \Rightarrow$ Landau thought the theory is inconsistent)

$\Rightarrow$ one can think of running coupling as of the virtual $q\bar{q}$ (or gg) pairs popping out of the vacuum & screening the color charge:

like molecules in a dielectric:



(I)
$\Rightarrow$ in QCD $\beta_2 > 0$ $\Rightarrow$



$\Rightarrow$ at large Q^2 / short distances ($\sim \hbar/Q \sim 1/Q$) the coupling is small!

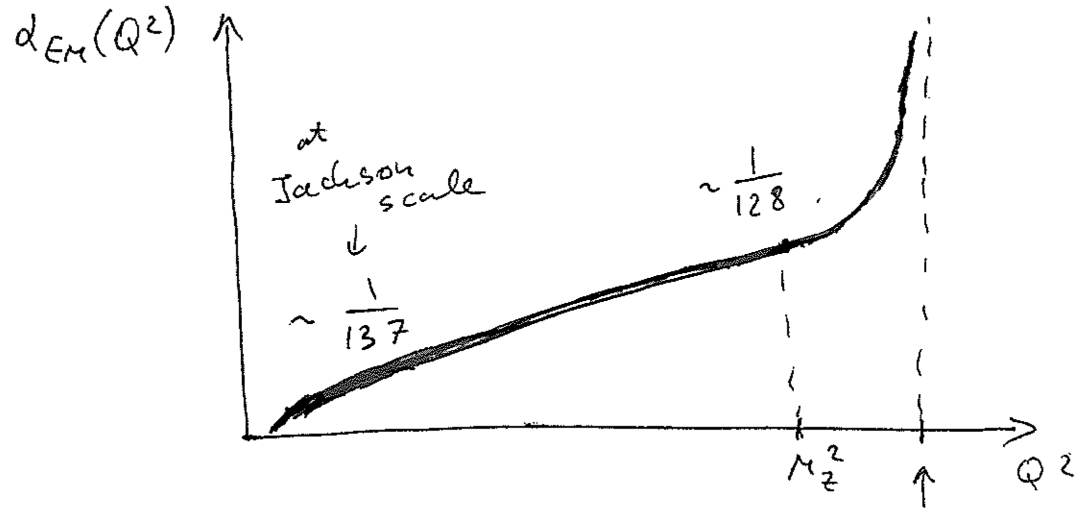
$\Rightarrow$ QCD at short distances is weakly coupled ~ quarks and gluons are asymptotically free! (Politzer, Gross, Wilczek '73)

(see attached plot)

II in QED $\beta_2^{QED} < 0 \Rightarrow$

$$\alpha_{EM}(Q^2) = \frac{\alpha_{EM\mu}}{1 + \alpha_{EM\mu}\underbrace{\beta_2^{QED}}_{= -\frac{1}{3\pi}} \ln\frac{Q^2}{\mu^2}} = \frac{\alpha_\mu}{1 - \frac{\alpha_\mu}{3\pi}\ln\frac{Q^2}{\mu^2}}$$

$$\Rightarrow \boxed{\alpha_{EM}(Q^2) = \frac{\alpha_\mu}{1 + \frac{\alpha_\mu}{3\pi}\ln\frac{\mu^2}{Q^2}}}$$ ~ increases with Q^2



$\Rightarrow$ no asymptotic freedom in QED!

$\Rightarrow$ also has a Landau pole, but at large momenta ~ there QED may map onto some more "fundamental" theory, eliminating Landau pole...

$\Rightarrow$ in QCD with massless quarks mesons are massless.

$\Rightarrow$ baryons have a mass. Consider proton (the lightest baryon).

proton mass: M_p ~ dimensionfull quantity.

$M_p = M_p(\alpha_\mu, \mu) = \mu\, f(\alpha_\mu)$ as μ is the only dimension full scale.

$$\mu^2 \frac{d}{d\mu^2} M_p = 0 \Rightarrow \left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu}\right) M_p = 0$$

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu}\right)\left[\mu\, f(\alpha_\mu)\right] = 0$$

$$\mu^2 \frac{\partial}{\partial \mu^2}(\mu) = \frac{1}{2}\mu \Rightarrow \left(\frac{1}{2} + \beta \frac{\partial}{\partial \alpha_\mu}\right) f(\alpha_\mu) = 0$$

$$\Rightarrow \frac{d f(\alpha_\mu)}{d\alpha_\mu} = -\frac{1}{2\beta(\alpha_\mu)} f(\alpha_\mu) \Rightarrow \frac{df}{f} = -\frac{d\alpha_\mu}{2\beta(\alpha_\mu)}$$

$$\Rightarrow \ln f(\alpha_\mu) - \ln f(\alpha_0) = -\frac{1}{2}\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} = -\frac{1}{2}\rho(\alpha, \alpha_0)$$

$\Rightarrow f(\alpha_\mu) = f(\alpha_0)\, e^{-\frac{1}{2}\rho(\alpha, \alpha_0)}$ and the proton's mass is