Last time: Running Coupling and Asymptotic Freedom (cont'd)

$$\mathcal{M}(g, \mu) \to \mathcal{M}(g_\mu, \mu)$$

an observable

coupling must be a ftn of UV cutoff μ. (need to rearrange pert. theory)

$$\boxed{\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu}\right] \mathcal{M}\left(Q^2/\mu^2, \alpha_\mu\right) = 0}$$

Callan-Symanzik equation ~ μ^2-independence of $\mathcal{M}$

Def.
$$\boxed{\beta(\alpha_\mu) = \mu^2 \frac{d\alpha_\mu}{d\mu^2}}$$
~ beta-function of a field th'y

$$\alpha_\mu = \frac{g_\mu^2}{4\pi}.$$

Def. Running coupling:

$$\boxed{\alpha(Q^2) \equiv \rho^{-1}\left(\ln \frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right)}$$

where $\rho(\alpha_\mu) = \displaystyle\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')}$.

$\alpha(Q^2)$ is μ^2-independent.

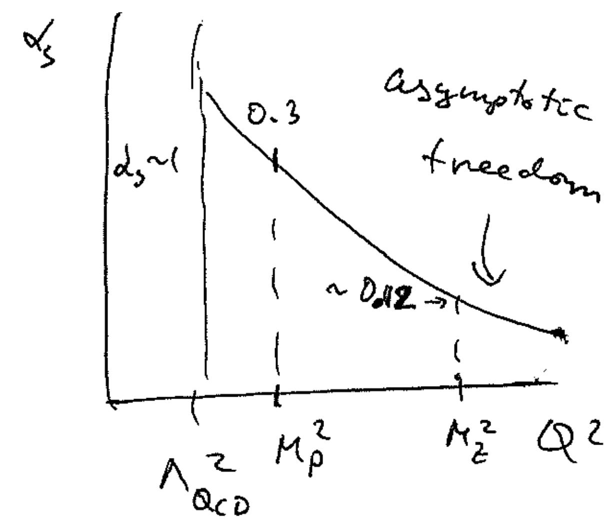
$$\mathcal{M}\left(Q^2/\mu^2, \alpha_\mu\right) \overset{\mu = Q}{=} \mathcal{M}(1, \alpha(Q^2)) = \mathcal{M}(\alpha(Q^2)) \text{ is also}$$

μ^2-independent

Any ftn. of $\alpha(Q^2)$ is μ^2-independent.

Last time: Finished talking about the running coupling. In QCD we observed that the 1-loop running coupling is

$$\alpha_s(Q^2) = \frac{\alpha_\mu}{1 + \alpha_\mu \beta_2 \ln \frac{Q^2}{\mu^2}}$$



$$\beta_2 = \frac{11 N_c - 2N_f}{12\pi}$$

$$\alpha_s(Q^2) = \frac{1}{\beta_2 \ln \frac{Q^2}{\Lambda^2_{QCD}}}$$

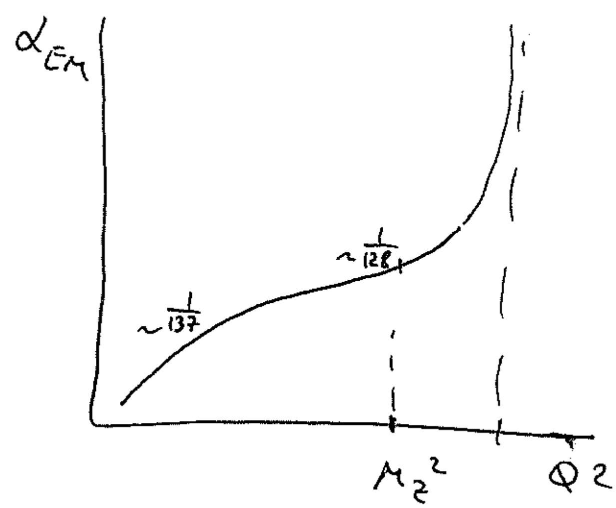
with $\Lambda_{QCD} \simeq 200\,\text{MeV}$ (fundamental scale of QCD)

$Q^2 = \Lambda^2_{QCD}$ ~ Landau pole

In QED:

$$\alpha_{EM}(Q^2) = \frac{\alpha_\mu}{1 - \frac{\alpha_\mu}{3\pi} \ln \frac{Q^2}{\mu^2}}$$

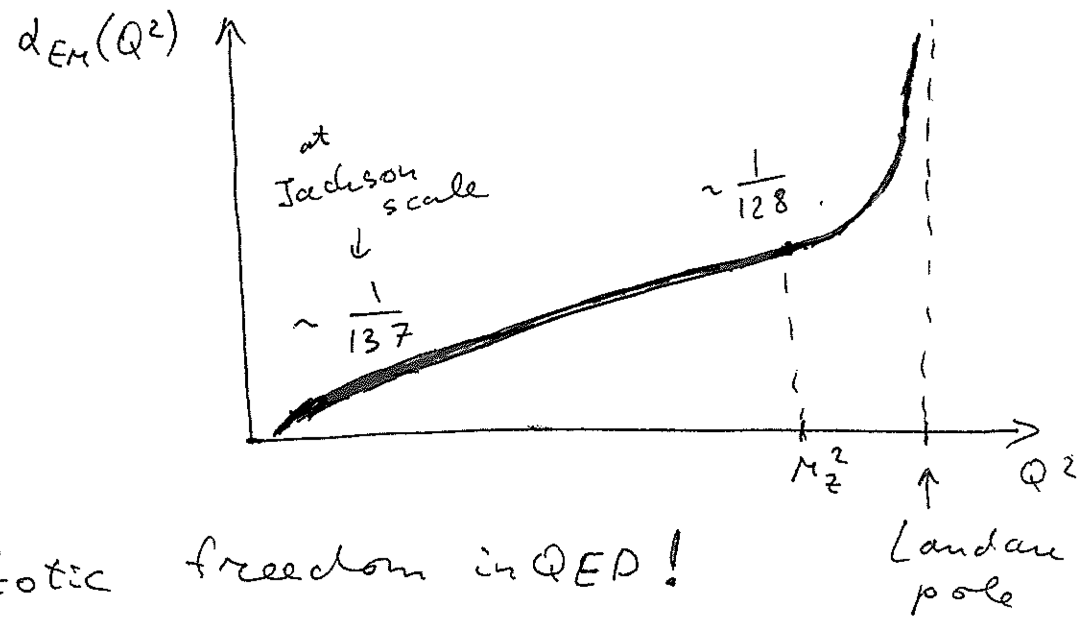
note the sign



II in QED $\beta_2^{QED} < 0 \Rightarrow$

$$\alpha_{EM}(Q^2) = \frac{\alpha_{EM\mu}}{1 + \alpha_{EM\mu} \underbrace{\beta_2^{QED}}_{= -\frac{1}{3\pi}} \ln \frac{Q^2}{\mu^2}} = \frac{\alpha_\mu}{1 - \frac{\alpha_\mu}{3\pi} \ln \frac{Q^2}{\mu^2}}$$

$$\Rightarrow \alpha_{EM}(Q^2) = \frac{\alpha_\mu}{1 + \frac{\alpha_\mu}{3\pi} \ln \frac{\mu^2}{Q^2}}$$ ~ increases with Q^2



$\Rightarrow$ no asymptotic freedom in QED!

$\Rightarrow$ also has a Landau pole, but at large momenta ~ there QED may map onto some more "fundamental" theory, eliminating Landau pole...

$\Rightarrow$ in QCD with massless quarks mesons are massless.

$\Rightarrow$ baryons have a mass. Consider proton (the lightest baryon).

proton mass: M_p ~ dimensionfull quantity.

$M_p = M_p(\alpha_\mu, \mu) = \mu\, f(\alpha_\mu)$ as μ is the only dimensionfull scale.

$$\mu^2 \frac{d}{d\mu^2} M_p = 0 \Rightarrow \left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu}\right) M_p = 0$$

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu}\right)\left[\mu f(\alpha_\mu)\right] = 0$$

$$\mu^2 \frac{\partial}{\partial \mu^2}(\mu) = \frac{1}{2}\mu \Rightarrow \left(\frac{1}{2} + \beta \frac{\partial}{\partial \alpha_\mu}\right) f(\alpha_\mu) = 0$$

$$\Rightarrow \frac{d f(\alpha_\mu)}{d\alpha_\mu} = -\frac{1}{2\beta(\alpha_\mu)} f(\alpha_\mu) \Rightarrow \frac{df}{f} = -\frac{d\alpha_\mu}{2\beta(\alpha_\mu)}$$

$$\Rightarrow \ln f(\alpha_\mu) - \ln f(\alpha_0) = -\frac{1}{2}\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} = -\frac{1}{2}\rho(\alpha, \alpha_0)$$

$$\Rightarrow f(\alpha_\mu) = f(\alpha_0)\, e^{-\frac{1}{2}\rho(\alpha, \alpha_0)}$$ and the

proton's mass is

$$\boxed{M_p = \mu\, f(\alpha_0)\, e^{-\frac{1}{2}\rho(\alpha_\mu, \alpha_0)}}$$

take $\beta(\alpha) = -\beta_2\alpha^2 \Rightarrow \rho(\alpha) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} = \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right)$

$$\Rightarrow M_p = \mu\, f(\alpha_0)\, e^{-\frac{1}{2\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right)}$$

M_p should not depend on α_0 (a cutoff) $\Rightarrow$

$\Rightarrow f(\alpha_0) \propto e^{-\frac{1}{2\beta_2}\frac{1}{\alpha_0}} \Rightarrow$ write $f(\alpha_0) = C_p e^{-\frac{1}{2\beta_2}\frac{1}{\alpha_0}}$

constant

$\Rightarrow \boxed{M_p = C_p \cdot \mu \cdot e^{-\frac{1}{2\beta_2}\frac{1}{\alpha_\mu}}}$ non-perturbative dependence on α_μ

$e^{-\frac{1}{x}}$ is a function $\neq$ to its Taylor series

$\Rightarrow$ non-perturbative!

Take $\beta(\alpha) = -\beta_2\alpha^2 - \beta_3\alpha^3 \Rightarrow$

pert. series

$$\rho(\alpha) = -\frac{1}{\beta_2}\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\alpha'^2\left(1 + \frac{\beta_3}{\beta_2}\alpha'\right)} = -\frac{1}{\beta_2}\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\alpha'^2}\left[1 - \frac{\beta_3}{\beta_2}\alpha' + \dots\right]$$

$$= \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right) + \frac{\beta_3}{\beta_2^2}\ln\frac{\alpha_\mu}{\alpha_0} + \dots$$

$$\Rightarrow M_p = \mu\, f(\alpha_0)\, e^{-\frac{1}{2}\left[\frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right) + \frac{\beta_3}{\beta_2^2}\ln\left(\frac{\alpha_\mu}{\alpha_0}\right) + \dots\right]}$$

$\Rightarrow$ pick $f(\alpha_0) = C_p\, e^{-\frac{1}{2\beta_2\alpha_0} - \frac{\beta_3}{2\beta_2^2}\ln\alpha_0}$

$\Rightarrow$ get $$M_p = C_p\, \mu\, \underbrace{e^{-\frac{1}{2\beta_2\alpha_\mu}}}_{\text{non-analytic ftn.}} (\alpha_\mu)^{-\frac{\beta_3}{2\beta_2^2}} \underbrace{(1 + o(\alpha_\mu))}_{\text{analytic function}}$$

$\Rightarrow$ can not calculate M_p in perturbation theory.

Finally, $M_p = C_p\, \mu\, e^{-\frac{1}{2\beta_2\alpha_\mu}}$, remember that $\alpha_\mu = \dfrac{1}{\beta_2 \ln \frac{\mu^2}{\Lambda_{QCD}^2}} \Rightarrow \dfrac{1}{2\beta_2\alpha_\mu} = \ln \dfrac{\mu}{\Lambda_{QCD}}$

$\Rightarrow M_p = C_p\, \mu \cdot e^{-\ln\frac{\mu}{\Lambda_{QCD}}} = C_p\, \Lambda_{QCD}$

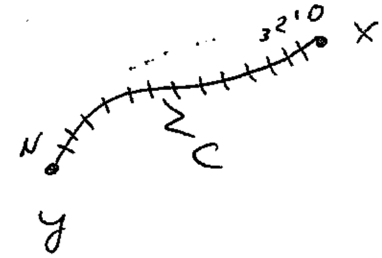
$\Rightarrow$ $M_p \sim \Lambda_{QCD}$ $\sim$ a non-perturbative QCD scale where the coupling α_s is large $\Rightarrow$ can't do perturbation theory there.

Wilson lines, loops & Heavy Quark Potential

Def. **Wilson line:**

$$\boxed{W_c(x,y) \equiv P_c \exp\left\{ ig \int_y^x dx'_\mu \, A^\mu(x') \right\}}$$

Def. where a path-ordered exponential is defined as follows. Cut the path C connecting y & x into slices (W_C depends on C!).



Then

$$\boxed{P_c \exp\left\{ ig \int_y^x dx'_\mu \, A^\mu(x') \right\} \equiv \lim_{N\to\infty} \prod_{i=1}^{N} \left[1 + ig\, \Delta x_i^\mu \, A_\mu(x_i) \right]}$$

right-most

left-most

$(x_0^\mu = x^\mu, \; x_N^\mu = y^\mu)$, $\Delta x_i^\mu = x_{i-1}^\mu - x_i^\mu$

Under gauge transform $A_\mu(x_i) \to S(x_i) A_\mu(x_i) S^{-1}(x_i) - \frac{i}{g}\left(\partial_\mu S(x_i)\right) S^{-1}(x_i)$

$$\Rightarrow W_c(x,y) \to \prod_{i=1}^{N} \left[1 + ig\, \Delta x_i^\mu \left(S(x_i) A_\mu(x_i) S^{-1}(x_i) - \frac{i}{g}\left(\partial_\mu S(x_i)\right) S^{-1}(x_i) \right) \right] =$$

use

$$S(x_{i-1}) = S(x_i) + \Delta x_i^\mu \, \partial_\mu S(x_i)$$

and neglect $o(\Delta x^2)$ terms in each factor

$$= \prod_{i=1}^{N} \left[1 + ig \left(\Delta x_i^{\mu}\, \mathcal{S}(x_{i-1})\, A_\mu(x_i)\, \mathcal{S}^{-1}(x_i) - \frac{i}{g} \left(\mathcal{S}(x_{i-1}) - \mathcal{S}(x_i) \right) \mathcal{S}^{-1}(x_i) \right) \right] = \prod_{i=1}^{N} \left[\cancel{1} + ig \left(\Delta x_i^{\mu}\, \mathcal{S}(x_{i-1}) A_\mu(x_i) \mathcal{S}^{-1}(x_i) - \frac{i}{g}\, \mathcal{S}(x_{i-1})\, \mathcal{S}^{-1}(x_i) + \cancel{\frac{i}{g}} \right) \right] = \prod_{i=1}^{N} \mathcal{S}(x_{i-1}) \cdot$$

$$\cdot \left[1 + ig\, \Delta x_i^{\mu} A_\mu(x_i) \right] \mathcal{S}^{-1}(x_i) = \mathcal{S}(x) \prod_{i=1}^{N} \left[1 + ig \Delta x_i^{\mu} A_\mu(x_i) \right] \cdot$$

$$\cdot\, \mathcal{S}^{-1}(y) = \mathcal{S}(x)\, W_c(x,y)\, \mathcal{S}^{-1}(y).$$

$$\Rightarrow \boxed{W_c(x,y) \longrightarrow \mathcal{S}(x) W_c(x,y)\, \mathcal{S}^{-1}(y)}$$

Def. Wilson loop:

$tr[W_c(x,x)]$ is called a <u>Wilson loop</u>.

(K. Wilson, '74?)

Under gauge transformation

$$tr[W_c(x,x)] \to tr[\mathcal{S}(x) W_c(x,x)\, \mathcal{S}^{-1}(x)] = tr[W_c(x,x)]$$

invariant! Wilson loop is gauge-invariant!

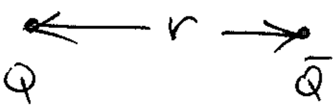
uses:

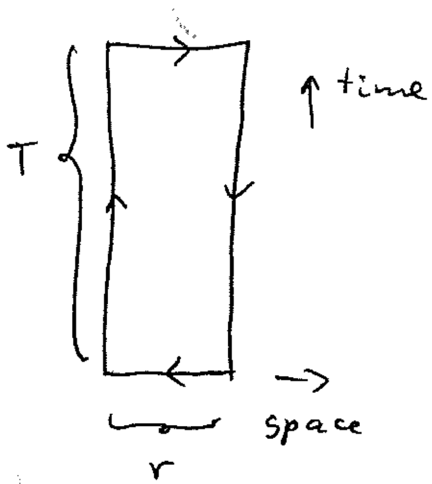
$\Rightarrow$ Wilson line represents quark propagator when one can neglect recoil. This works in high energy scattering and for static heavy quarks.

$\Rightarrow$ Wilson lines form links which can be used to define QCD action on the lattice for numerical simulations.

Heavy Quark Potential:

Suppose one wants to find heavy $Q\bar{Q}$ potential in QCD. How does one define the potential $V(r)$ in a gauge-invariant way?





Take a Wilson loop defined as shown.

$$\langle W \rangle \Big|_{T\to\infty} \simeq e^{-i\,T\,V(r)}$$

neglect interaction with gauge links.
(it does not scale with T to the same degree)

$$V(r) = \lim_{T\to\infty} \left[\frac{i}{T} \ln \langle W \rangle \right].$$

~ can calculate numerically on the lattice

Note that, since Feynman path integral time-ordery operators, one can write

$$\boxed{tr\left[W_c(x,x)\right] = \frac{\int \mathcal{D}A_\mu \, e^{i S[A_\mu]} \cdot e^{ig \int j^a_\mu(x) A^{a\mu}(x) d^4x}}{\int \mathcal{D}A_\mu \, e^{i S[A_\mu]}}}$$

where $j^a_\mu(x)$ is some external current, which is non-zero only along the contour C.

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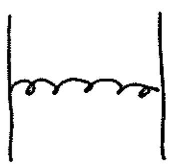
r is the only scale in $V(r) \Rightarrow \alpha_s = \alpha_s(1/r^2)$

if $r \ll \frac{1}{\Lambda_{QCD}} \Rightarrow \alpha_s\left(\frac{1}{r^2}\right) \ll 1 \Rightarrow$ can use perturbative QCD

The potential is (see pp. 38-40 of these notes)



$$\boxed{V(r)\Big|_{r \ll \frac{1}{\Lambda_{QCD}}} \simeq -\frac{\alpha_s C_F}{r} = -\frac{4}{3}\frac{\alpha_s}{r}}$$

$\Rightarrow$ this is a Coulomb-like potential, similar to classical E&M.