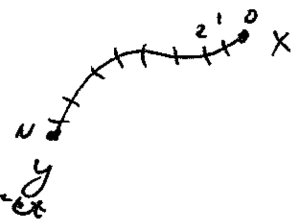
<u>Last time:</u> <u>Wilson lines, loops & Heavy Quark Potential</u>

(cont'd)

Wilson line

$$\boxed{W_c(x,y) = P_c \exp\left\{ ig \int_y^x dx'_\mu \, A^\mu(x') \right\}}$$



← right-most in the product

$$\boxed{W_c(x,y) = \lim_{N\to\infty} \prod_{i=1}^{N} \left[1 + ig\, \Delta x_i^\mu A_\mu(x_i) \right]}$$

← left-most

$P_c \exp$ ~ path-ordered exponential

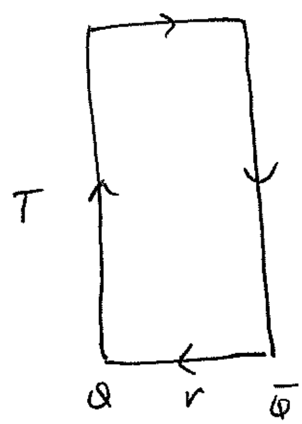
Under gauge transformations:

$$\boxed{W_c(x,y) \to S(x)\, W_c(x,y)\, S^{-1}(y)} \Rightarrow$$

Wilson loops: $\mathrm{tr}\{W_c(x,x)\}$ gauge-invariant

<u>Heavy Quark Potential</u>:

$$\boxed{V(r) = \lim_{T\to\infty} \left[\frac{i}{T} \ln \langle W(r,T) \rangle \right]}$$



$$r \ll \frac{1}{\Lambda_{QCD}} \Rightarrow V(r) \simeq -\frac{\alpha_s C_F}{r}$$

$$r \gg \frac{1}{\Lambda_{QCD}} \Rightarrow V(r) \simeq \sigma r, \quad \sigma \simeq 1 \frac{\text{GeV}}{\text{fm}}$$ ~ string tension

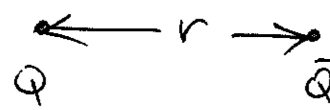
uses:

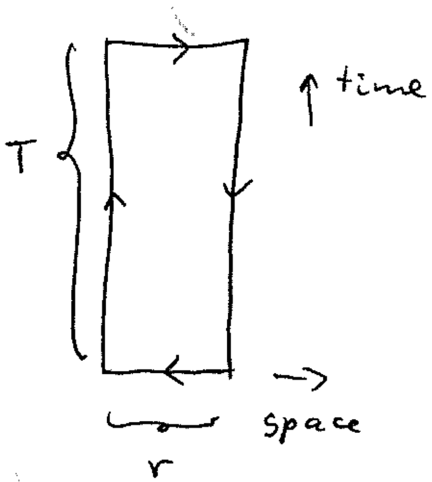
$\Rightarrow$ Wilson line represents quark "propagator" ← includes all gluon interax. when one can neglect recoil. This works in high energy scattering and for static heavy quarks.

$\Rightarrow$ Wilson lines form links which can be used to define QCD action on the lattice for numerical simulations.

Heavy Quark Potential:

Suppose one wants to find heavy $Q\bar{Q}$ potential in QCD. How does one define the potential $V(r)$ in a gauge-invariant way?





Take a Wilson loop defined as shown.

$$\langle W \rangle \Big|_{T \to \infty} \simeq e^{-i T V(r)}$$

↑ neglect interaction with gauge links. (it does not scale with T to the same degree)

$$V(r) = \lim_{T \to \infty} \left[\frac{i}{T} \ln \langle W \rangle \right].$$

~ can calculate numerically on the lattice

Note that, since Feynman path integral time-ordered operators, one can write

$$\boxed{\mathrm{tr}\left[W_c(x,x)\right] = \frac{\int \mathcal{D}A_\mu \, e^{iS[A_\mu]} \cdot e^{ig\int j^a_\mu(x) A^{a\mu}(x) d^4x}}{\int \mathcal{D}A_\mu \, e^{iS[A_\mu]}}}$$

where $j^a_\mu(x)$ is some external current, which is non-zero only along the contour C.

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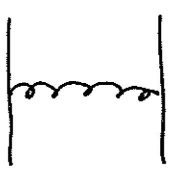
r is the only scale in $V(r) \Rightarrow \alpha_s = \alpha_s(1/r^2)$

if $r \ll \frac{1}{\Lambda_{QCD}} \Rightarrow \alpha_s\left(\frac{1}{r^2}\right) \ll 1 \Rightarrow$ can use perturbative QCD

The potential is (see pp. 38-40 of these notes)

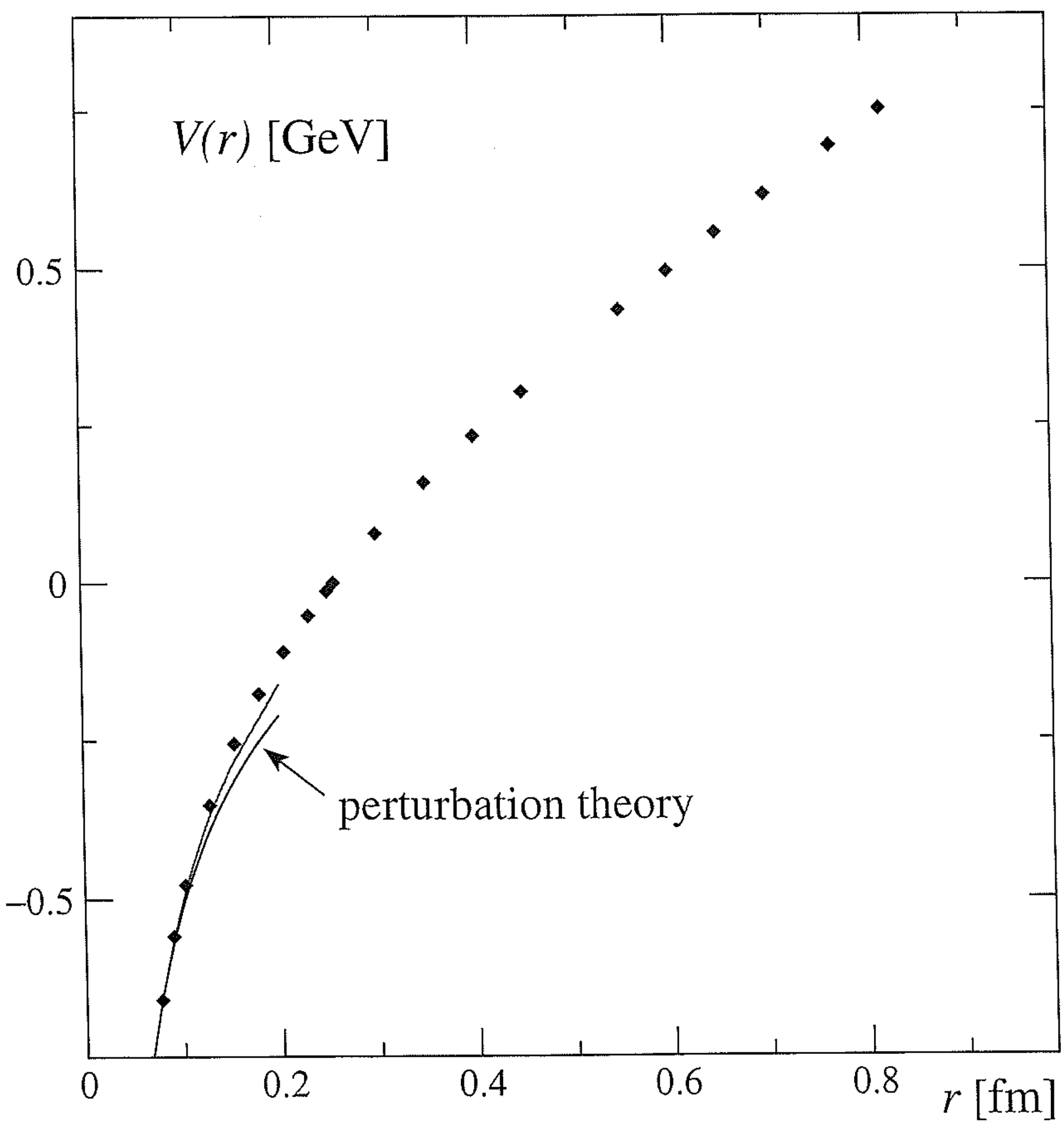


$$\boxed{V(r)\Big|_{r \ll \frac{1}{\Lambda_{QCD}}} \simeq -\frac{\alpha_s C_F}{r} = -\frac{4}{3}\frac{\alpha_s}{r}}$$

$\Rightarrow$ this is a Coulomb-like potential, similar to classical E&M.

Lattice QCD: data points

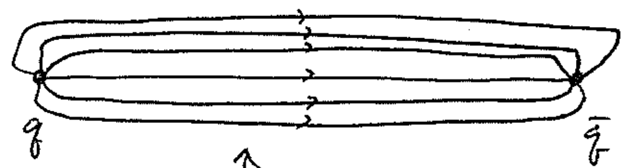
perturbative QCD: solid lines + band.



Longer Distances: $r\Lambda_{QCD} \gtrsim 1 \Rightarrow$

$\alpha_s = \alpha_s\left(\frac{1}{r^2}\right) \sim \alpha_s(\Lambda_{QCD}^2) \sim 1 \Rightarrow$ perturbative approach breaks down as α_s is not small anymore!

Qualitative picture of what happens: draw force lines as:



$\sim$ constant force in-between, inside the flux tube

$$\Rightarrow \quad V(r) \propto \underset{\text{force}}{F} \cdot r \quad \Rightarrow \quad V(r)\Big|_{r\Lambda_{QCD} \gg 1} \simeq \sigma r$$

dimensions of σ ~ mass squared, $\sigma \simeq \Lambda_{QCD}^2$.

$\Rightarrow$ think of a flux tube as a relativistic string: σ is string tension:

$$\sigma \simeq 1\ \frac{GeV}{fm} \simeq \frac{1}{5}\ GeV^2$$

Relativistic particle: the action is proportional to proper time τ, such that

$$S_{\text{particle}} = -mc^2 \int d\tau .$$

Relativistic string: the action is proportional to "proper area" of a world-sheet:

$$S_{\text{string}} = - \underset{\substack{\uparrow \\ \text{string tension}}}{\sigma} \cdot (\text{Area}).$$

(put $c=1$ for simplicity).

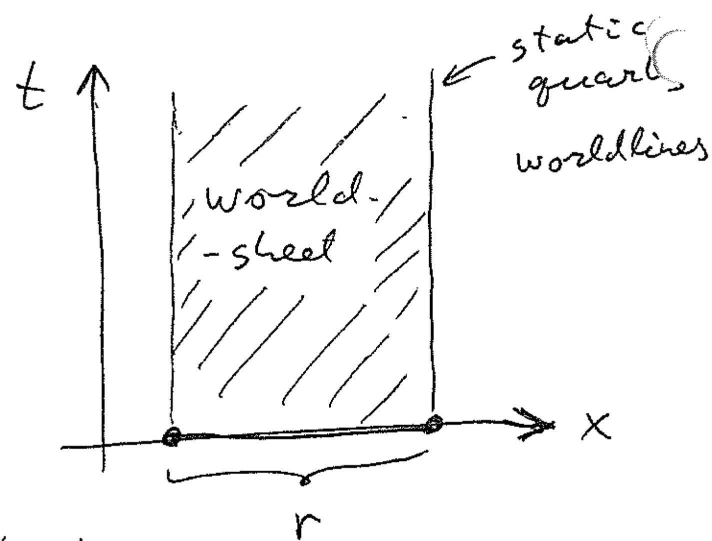
e.g. Nambu-Goto action:

$$S = -\frac{1}{2\pi\alpha'}\int d\sigma\, d\tau \sqrt{-\det g_{\alpha\beta}}$$

$$g_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta}$$

$$\xi^1 = \tau,\ \xi^2 = \sigma$$

Consider a static string between 2 quarks:



to find classical configuration need to extremize the action S_{string} => minimize the area of string worldsheet

=> obviously min. is achieved for straight string with the action $S^{\text{classical}}_{\text{string}} = -\sigma \cdot \int dt \cdot \int_0^r dx$

$$= -\sigma \int dt \cdot r = \int dt \cdot L = \int dt\, (\underset{\substack{\| \\ 0 \\ \text{as no motion}}}{K} - V(r)) = \int dt\, [-V(r)]$$

$\Rightarrow$ $V(r) = \sigma r$ as desired!

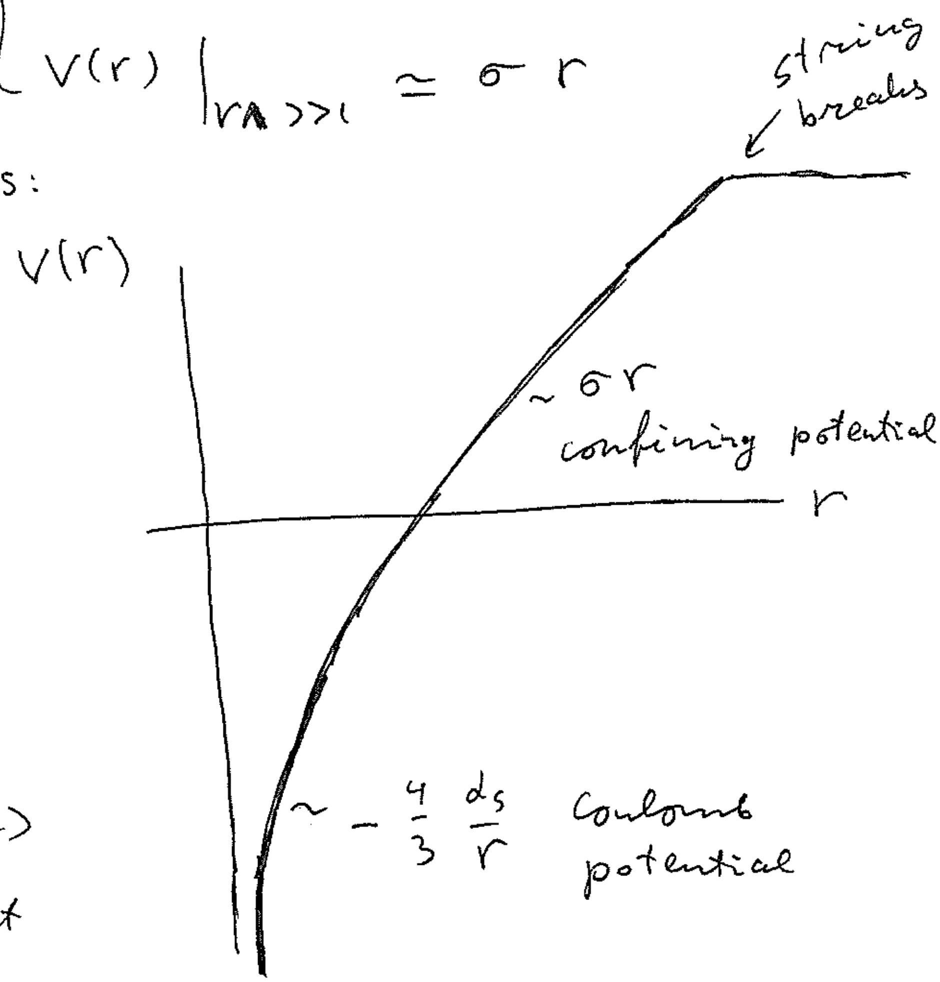
(note the difference from non-relativistic string in classical mechanics which has $V(r) \sim \frac{1}{2} k r^2 \Rightarrow$ force $= kr$)

$\Rightarrow$ the attractive force is <u>constant</u>: $F = \sigma$.

$\Rightarrow$ We know that
$$\begin{cases} V(r)\big|_{r\Lambda \ll 1} \simeq -\frac{4}{3}\frac{\alpha_s}{r} \\ V(r)\big|_{r\Lambda \gg 1} \simeq \sigma r \end{cases}$$

The full potential is:

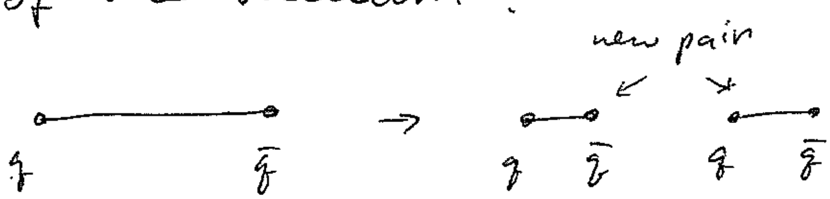
(see attached lattice data handout)



Linear potential is <u>confining</u>: quarks can not escape.

If string breaks $\Rightarrow$

$\Rightarrow$ get $q\bar{q}$ pair out of the vacuum:



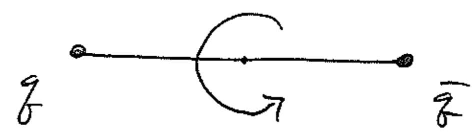
Good interpolation:

$$\boxed{V(r) \simeq -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r}$$

"Cornell potential".

String model works amazingly well: think of $q\bar{q}$ state as a meson. If the meson has spin $\Rightarrow$ think of an ultra-relativistic rotating string:

d ~ string length



if q & $\bar{q}$ rotate with velocity $= 1$ (UR quarks) $\Rightarrow$ $v = \frac{r}{d/2} = \frac{2r}{d}$.

r ~ distance from string element to rot. center

v ~ velocity of string element.

$$\Rightarrow M = \int \frac{dm}{\sqrt{1-v^2}} = 2\int_0^{d/2} \frac{\sigma\, dr}{\sqrt{1-v^2}} = 2\sigma \cdot \int_0^{d/2} \frac{dr}{\sqrt{1-\left(\frac{2r}{d}\right)^2}} = 2\sigma \cdot \frac{d}{2} \cdot \underbrace{\int_0^1 \frac{d\xi}{\sqrt{1-\xi^2}}}_{\substack{\pi/2 \\ \shortparallel \\ \left(\arcsin \xi\right)\Big|_0^1}} = \frac{\pi}{2}\sigma d.$$

The angular momentum (meson's spin)

is
$$J = \int \frac{r v \, dm}{\sqrt{1-v^2}} = 2\sigma \int_0^{d/2} \frac{r v \, dr}{\sqrt{1-v^2}} =$$

$$= 2\sigma \int_0^{d/2} \frac{dr \cdot (2r/d) \cdot r}{\sqrt{1-\left(\frac{2r}{d}\right)^2}} = 2\sigma \left(\frac{d}{2}\right)^2 \underbrace{\int_0^1 d\xi \, \frac{\xi^2}{\sqrt{1-\xi^2}}}_{\pi/4} =$$

$$= \frac{\sigma d^2}{2} \cdot \frac{\pi}{4} = \frac{\pi \cdot \sigma d^2}{8} .$$

$\Rightarrow$ meson mass $\quad \boxed{M = \frac{\pi}{2} \sigma d}$ $\quad$ Gasiorowicz & Rosner '81

meson spin $\quad \boxed{J = \frac{\pi \sigma d^2}{8}}$

$$\Rightarrow \quad J = \frac{\pi}{8} \sigma \cdot \left(\frac{2M}{\pi \sigma}\right)^2 = \frac{1}{2\pi\sigma} M^2$$

$\Rightarrow$ $\boxed{J = \frac{1}{2\pi\sigma} M^2}$ is an example of a Regge trajectory

In general, on the basis of phenomenological evidence, people noticed that

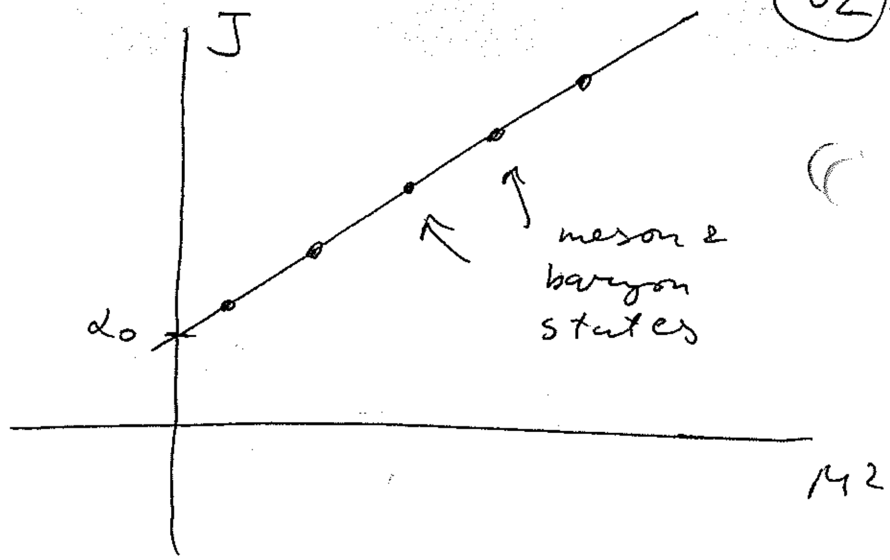
$$\boxed{J = \alpha_0 + \alpha' M_J^2}$$

Chew & Frautschi '61

$\alpha_0 \sim$ intercept

$\alpha' \sim$ slope

Regge trajectory:



we got $\alpha' = \frac{1}{2\pi\sigma}$

or $\sigma = \frac{1}{2\pi\alpha'}$.

using $\sigma = 1\ \mathrm{GeV/fm}$

$$\alpha' = \frac{1}{2\pi\sigma} \approx \frac{5}{2\pi}\ \mathrm{GeV}^{-2}$$

experimentally $\alpha' \simeq 0.25\ \mathrm{GeV}^{-2}$.

$\Rightarrow$ successes of string approximation to strong interaction data led to proposal of string theory as the theory of strong interactions in the '60's.

$\Rightarrow$ that idea was killed by $e^+e^- \to$ hadrons & (mostly) DIS data & string theory moved on to gravity in '84.

R. Alkofer, J. Greensite, J. Phys. G. 34 (2007) S3

hep-ph/0610365

see also G. Bali, hep-ph/0001312 Phys. Rept. 343, p.1 (2001)

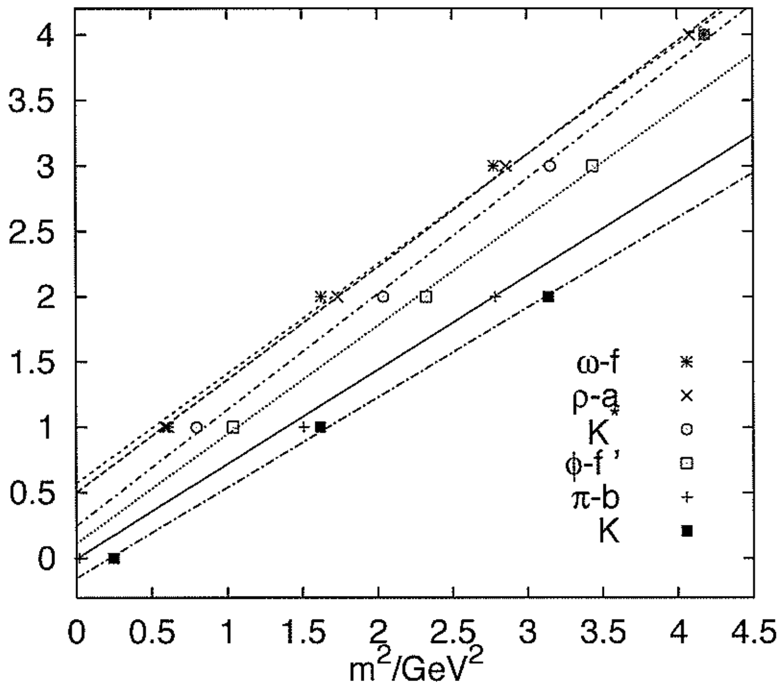


Figure 1. Regge trajectories for the low-lying mesons (adapted from ref. [2]).

2.1. Regge Trajectories and the Spinning Stick Model

A remarkable pattern emerges in the hadronic spectrum, when the spin of mesons (and baryons) is plotted against their squared mass, as shown in Fig. 1. In such plots the mesons and baryons of given flavor quantum numbers seem to lie on nearly parallel straight lines, known as linear Regge trajectories. This is a very striking feature of the hadronic spectrum, nothing similar is found in the electroweak theory, and the question is why it occurs.

Suppose that we picture a meson as a straight line of length $L = 2R$, with mass per unit length σ. The line rotates about a perpendicular axis through its midpoint, such that the endpoints of the line are moving at the speed of light, $v(R) = c = 1$. Then for the energy in the rest frame, *i.e.* the mass, of the spinning stick we have

$$m = \text{Energy} = 2\int_0^R \frac{\sigma dr}{\sqrt{1-v^2(r)}} = 2\int_0^R \frac{\sigma dr}{\sqrt{1-r^2/R^2}} = \pi\sigma R\,, \tag{3}$$

and for the angular momentum

$$J = 2\int_0^R \frac{\sigma r v(r) dr}{\sqrt{1-v^2(r)}} = \frac{2}{R}\int_0^R \frac{\sigma r^2 dr}{\sqrt{1-r^2/R^2}} = \tfrac{1}{2}\pi\sigma R^2\,. \tag{4}$$

Comparing the two expressions, we see that

$$J = \frac{1}{2\pi\sigma} m^2 = \alpha' m^2 \tag{5}$$

The constant α' is known as the *"Regge slope"*.

From the data one estimates $\alpha' = 1/(2\pi\sigma) = 0.9 \text{ GeV}^{-2}$, which gives a mass/unit length of the string, or *"string tension"*, of

$$\sigma \approx 0.18 \text{ GeV}^2 \approx 0.9 \text{ GeV/fm}. \tag{6}$$

The spinning stick model is, of course, only a caricature of the real situation. In fact the various Regge trajectories do not pass through the origin, and have slightly different slopes. To make the model more realistic, one might want to relax the requirement of rigidity, and allow the "stick" to fluctuate in transverse directions. This line of thought leads to string theory. However, since QCD is the theory of quarks and gluons, the question to be answered is how a stick-like or string-like object actually emerges from that theory.

One possible answer is via the formation of a color electric flux tube. We imagine that the color electric field running between a static quark and antiquark is, for some reason, squeezed into a cylindrical region, whose cross-sectional area is nearly constant as quark-antiquark separation L increases. In that case, the energy stored in the color electric field will grow linearly with quark separation, *i.e.*

$$\text{Energy} = \sigma L \quad \text{with} \quad \sigma = \int d^2x_\perp \, \tfrac{1}{2}\vec{E}^a \cdot \vec{E}^a \tag{7}$$

where the integration is over a cross-section of the flux tube. This means that there will be a linearly rising potential energy associated with static sources (the "static quark potential"), and an infinite energy is required to separate these charges an infinite distance.

In this way the pattern of metastable states in the hadron spectrum suggests a picture of how the color electric field energy, in the absence of light quark pair creation, would grow with quark separation.

2.2. *Wilson Loops and Lattice Simulations*

The most reliable evidence we have about the static quark potential is obtained from computer simulations of quantum chromodynamics. For this purpose it is useful to simulate a version of QCD in which the quarks are very massive, and pair creation in the vacuum can be ignored.

Let $Q(t)$ be the creation operator of a state at time t containing a very massive quark and a very massive antiquark, separated by a distance R. There are many operators of that sort, but, unless we fix a gauge, it is necessary for Q to be gauge-invariant. If not, then Q and correlators of Q will simply average to zero in the functional integral over gauge fields A and the quark fields ψ. Consider the unequal-times correlator

$$\langle Q^\dagger(T)Q(0)\rangle = \frac{1}{Z}\int DAD\psi D\overline{\psi}\, Q^\dagger(T)Q(0)\, e^{iS} = \langle\Psi_0|Q^\dagger e^{-i(H-\mathcal{E}_0)T}Q|\Psi_0\rangle \tag{8}$$

where H is the Hamiltonian operator, $\mathcal{E}_0$ is the vacuum energy and Ψ_0 is the vacuum state, in any gauge (the gauge choice does not matter if Q is gauge invariant). By transforming the theory from Minkowski space to Euclidean space by a Wick rotation of the time coordinate $t \to it$, and inserting a complete set of energy eigenstates $\{\Psi_n\}$ with the quantum numbers of the heavy quark-antiquark pair, the above expression becomes