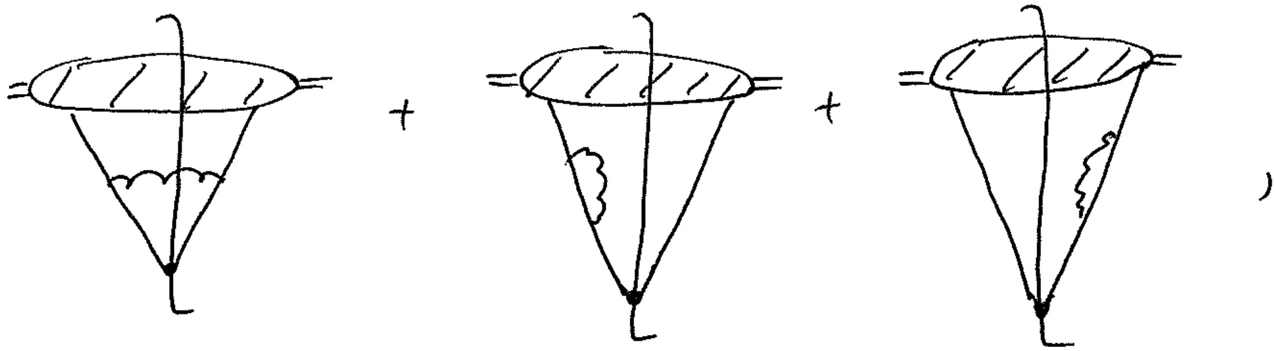
Last time | We completed the calculation of the 1-loop corrections to quark PDF,



and obtained an equation,

$$\frac{\partial}{\partial \ln Q^2} q^f(x, Q^2) = \frac{\alpha_s}{2\pi} \int\limits_x^1 \frac{dx_1}{x_1} P_{qq}\left(\frac{x_1}{x}\right) q^f(x_1, Q^2).$$

Here $P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z}\right)_+$ is the quark-quark <u>splitting function</u>.

$$\int\limits_0^1 dz \, [h(z)]_+ \, f(z) = \int\limits_0^1 dz \, h(z) \, [f(z) - f(1)]$$

"plus notation"

where $x = \frac{k^+}{p^+}$, $x_1 = \frac{k_1^+}{p^+}$ $\Rightarrow z = \frac{k^+}{k_1^+} = \frac{x}{x_1}$.

as $z < 1 \Rightarrow x_1 > x$ in the integral.

Including the virtual terms (B and C)

gives

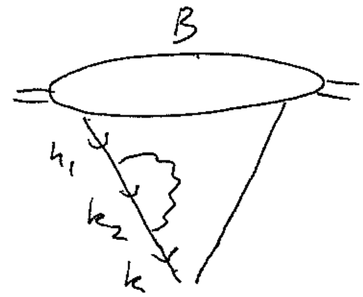
$$P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z}\right)_+$$

~ splitting function

where

$$\int_0^1 dz \, [h(z)]_+ \, f(z) = \int_0^1 dz \, h(z) \, [f(z) - f(1)]$$

Easy to understand:



$\propto \delta(k - k_1)$

$\Downarrow$

$x = x_1$

$\Rightarrow q_f(x_1, Q^2) = q_f(x, Q^2)$

as $x = x_1$

"real" part, diagram A

$$Q^2 \frac{\partial}{\partial Q^2} q^f(x, Q^2) = \frac{\alpha C_F}{2\pi} \left[\int_x^1 \frac{dz}{z} \frac{1+z^2}{1-z} \cdot q^f\left(\frac{x}{z}, Q^2\right) - \right.$$

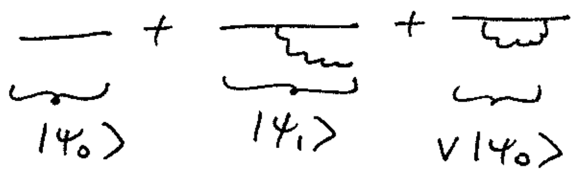
$$\left. - \int_0^1 dz \frac{1+z^2}{1-z} q^f(x, Q^2) \right]$$

virtual corrections, graphs B & C

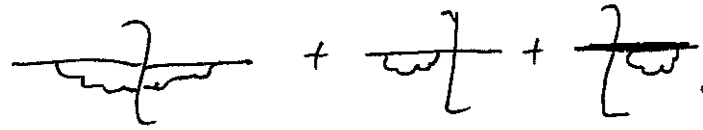
$z \to 1$ divergence is cancelled between the real (A) and virtual (B+C) terms.

bare quark state $|\psi_0\rangle$ = —— $\Rightarrow \langle\psi_0|\psi_0\rangle = 1$

(normalization)

dressed quark state $|\psi\rangle$ = 

normalization:

$\langle\psi|\psi\rangle = 1 = \langle\psi_0|\psi_0\rangle +$ 

$= 1 + \langle\psi_1|\psi_1\rangle + 2V\langle\psi_0|\psi_0\rangle = 1 + \langle\psi_1|\psi_1\rangle + 2V$

$\Rightarrow \boxed{V = -\frac{1}{2}\langle\psi_1|\psi_1\rangle}$

$\Rightarrow$ graphs $B, C = -\frac{1}{2} A \Rightarrow \boxed{B + C = -A}$.

~ simply imposed probability conservation!

$$\frac{\partial}{\partial \ln Q^2} q^f(x, Q^2) = \frac{\alpha C_F}{2\pi} \int_x^1 \frac{dx_1}{x_1} \left(\frac{1+\left(\frac{x}{x_1}\right)^2}{1-\frac{x}{x_1}} \right)_+ q^f(x_1, Q^2) =$$

$$= \left| \begin{array}{l} z = \frac{x}{x_1} \\ \frac{dx_1}{x_1} = -\frac{dz}{z} \end{array} \right. = \frac{\alpha C_F}{2\pi} \int_x^1 \frac{dz}{z} \left(\frac{1+z^2}{1-z} \right)_+ q^f\left(\frac{x}{z}, Q^2\right)$$

$$= \frac{\alpha C_F}{2\pi} \int_0^1 dz \underbrace{\left(\frac{1+z^2}{1-z} \right)_+}_{h(z)} \underbrace{\frac{1}{z} q^f\left(\frac{x}{z}, Q^2\right) \theta(z-x)}_{f(z)}$$

$$= \left| \begin{array}{l} \text{using} \\ \int_0^1 dz \left[h(z)\right]_+ f(z) = \int_0^1 dz\, h(z) \left[f(z) - f(1)\right] \end{array} \right.$$

$$= \frac{\alpha C_F}{2\pi} \int_0^1 dz \frac{1+z^2}{1-z} \left[\frac{1}{z} q^f\left(\frac{x}{z}\right) \theta(z-x) - q^f(x) \underset{1}{\underset{\shortparallel}{\theta(1-x)}} \right]$$

$$= \frac{\alpha_s C_F}{2\pi} \left[\int_x^1 \frac{dz}{z} \frac{1+z^2}{1-z} q^f\left(\frac{x}{z}, Q^2\right) - \int_0^1 dz \frac{1+z^2}{1-z} q^f(x) \right]$$

(Def.) Defining flavor singlet distribution

$$\Sigma(x, Q^2) \equiv \sum_f \left[q^f(x, Q^2) + q^{\bar{f}}(x, Q^2) \right]$$

(Def.) and flavor non-singlet

$$\Delta^{f\bar{f}}(x, Q^2) \equiv q^f(x, Q^2) - q^{\bar{f}}(x, Q^2)$$

we write

$$Q^2 \frac{\partial}{\partial Q^2} \Delta^{f\bar{f}}(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dx_1}{x_1} P_{qq}\left(\frac{x}{x_1}\right) \cdot \Delta^{f\bar{f}}(x_1, Q^2)$$

and

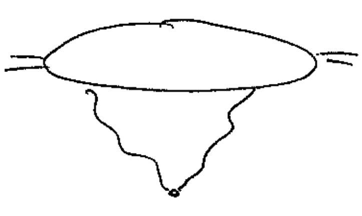
$$Q^2 \frac{\partial}{\partial Q^2} \begin{pmatrix} \Sigma(x, Q^2) \\ G(x, Q^2) \end{pmatrix} = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dx_1}{x_1} \begin{pmatrix} P_{qq}\left(\frac{x}{x_1}\right) & P_{qG}\left(\frac{x}{x_1}\right) \\ P_{Gq}\left(\frac{x}{x_1}\right) & P_{GG}\left(\frac{x}{x_1}\right) \end{pmatrix} \cdot \begin{pmatrix} \Sigma(x_1, Q^2) \\ G(x_1, Q^2) \end{pmatrix}$$

Dokshitzer, Gribov, Lipatov, Altarelli, Parisi

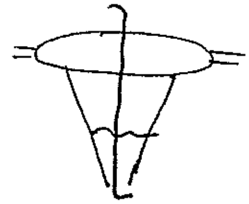
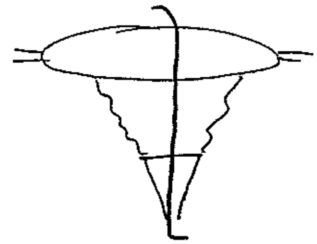
(DGLAP) Equations

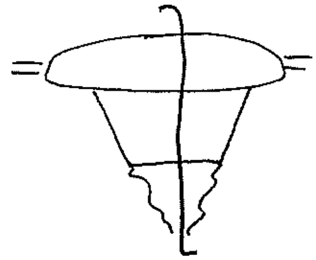
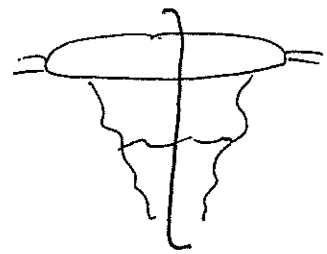
GL ~ QED case ~ '72

D, A & P ~ QCD case, '77

(Def.) $G(x, Q^2) =$  $\sim \langle A_i A_i \rangle$ in $A_+ = 0$ gauge

↖ gluon distribution function

$P_{qq} \sim$  , $P_{qG} =$ 

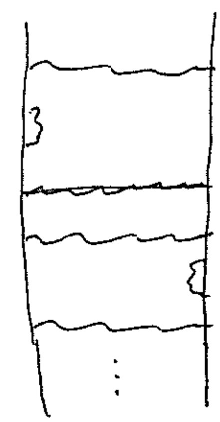
$P_{Gq} \sim$  , $P_{GG} \sim$ 

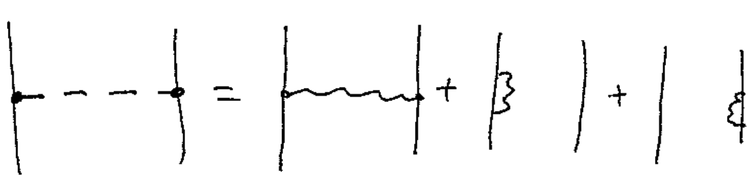
After explicit calculations one gets the <u>splitting functions</u>:

$$\begin{cases} P_{qq}(z) = C_F \left(\dfrac{1+z^2}{1-z} \right)_+ \\ P_{Gq}(z) = C_F \, \dfrac{1+(1-z)^2}{z} \\ P_{qG}(z) = N_f \left[z^2 + (1-z)^2 \right] \\ P_{GG}(z) = 2N_c \left[\dfrac{z}{(1-z)_+} + \dfrac{1-z}{z} + z(1-z) \right] + \dfrac{11 N_c - 2N_f}{6} \delta(z-1) \end{cases}$$

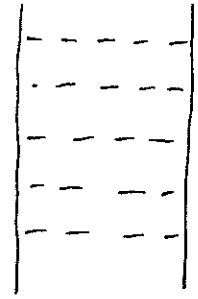
Note that $P_{Gq}(z)$ can be obtained from $P_{qq}(z)$ by substituting $z \to 1-z$ and dropping virtual corrections.

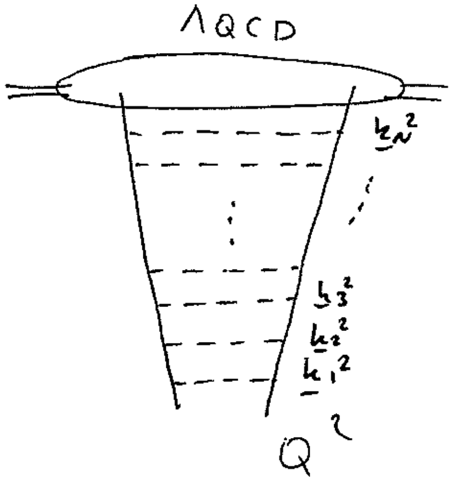
Iterate the evolution for $q_f(x, Q^2)$:



Defining 

We get a ladder diagram:



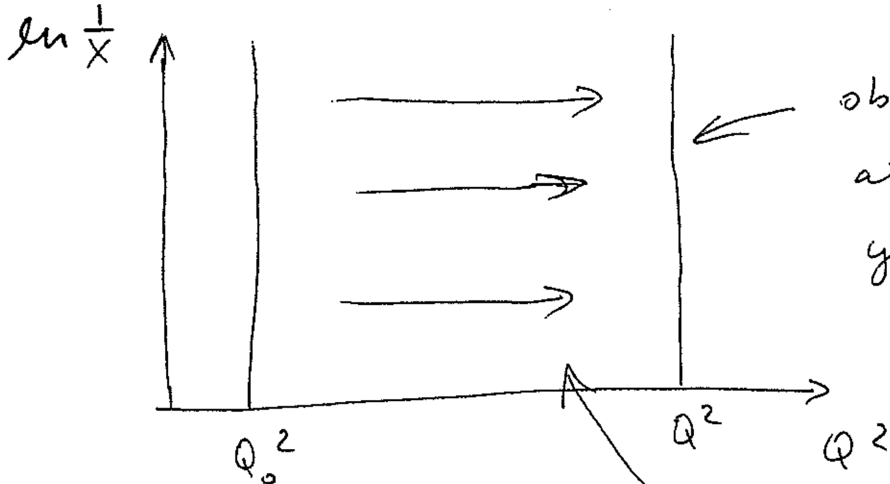


$Q^2 \gg k_1^2 \gg k_2^2 \gg \ldots \gg k_N^2 \gg \Lambda_{QCD}^2$

DGLAP resums ladder graphs with the ladder connecting scales Q^2 and Λ_{QCD}^2, $Q^2 \gg \Lambda_{QCD}^2$

such that $\ln Q^2/\Lambda_{QCD}^2 \gg 1$ and $\alpha_s \ln \frac{Q^2}{\Lambda_{QCD}^2} \sim 1$ is the resummation parameter.

How does DGLAP work?



obtain distribution at whatever Q^2 you wanted.

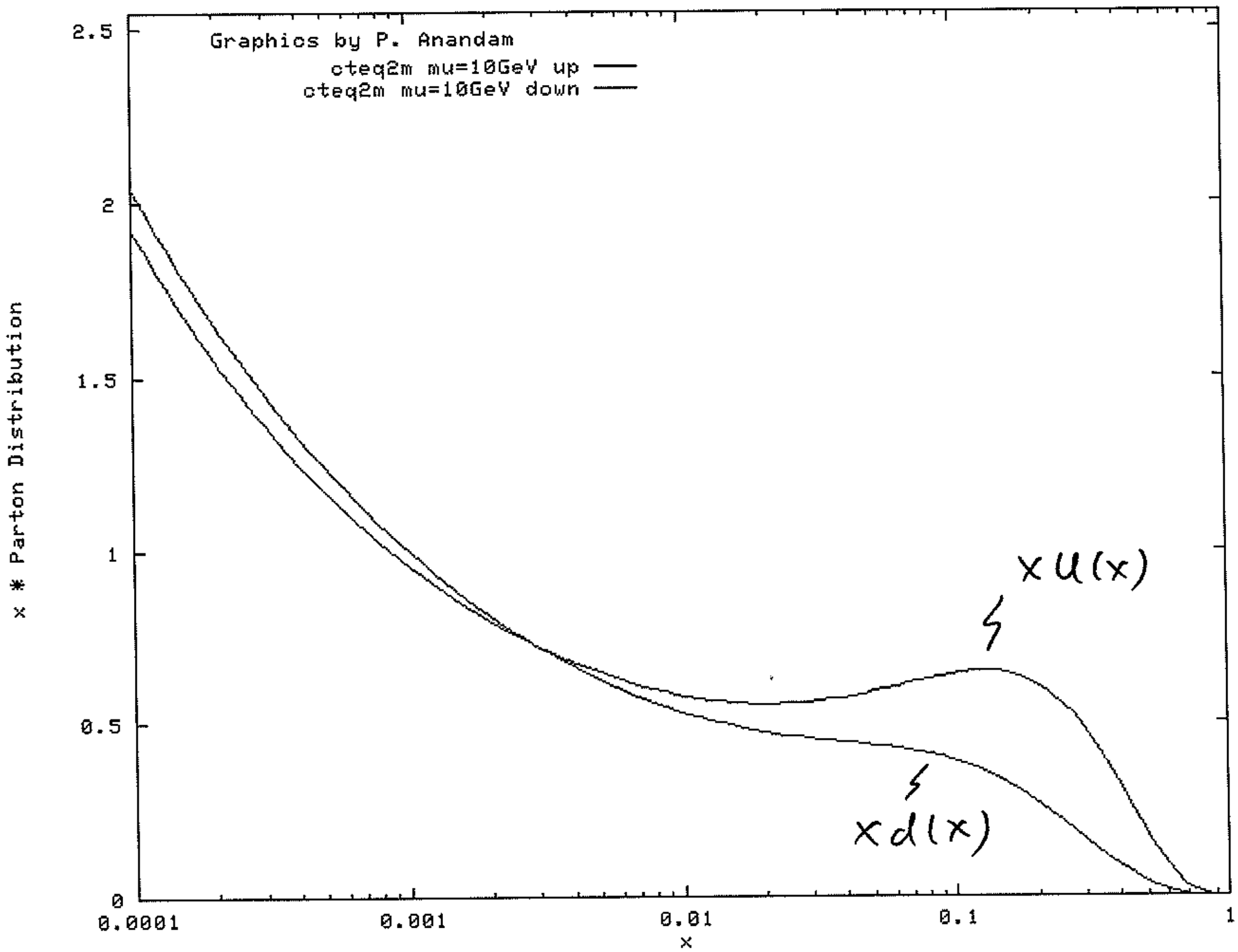
start with some initial condition = function $q^f(x, Q_0^2)$

Solve the DGLAP equations ("evolve" the distribution function)

=> people calculate PDF's (Parton Distribution Functions) & fit the data. See attachements for PDF examples.

Parton Distribution Graph

(Number of graphs plotted since 21 November 2000: 658)



$Q = 10$ GeV $\Rightarrow Q^2 = 100$ GeV^2

at large-x valence quarks dominate

$\Rightarrow x u_v(x) = 2\, x d_v(x)$

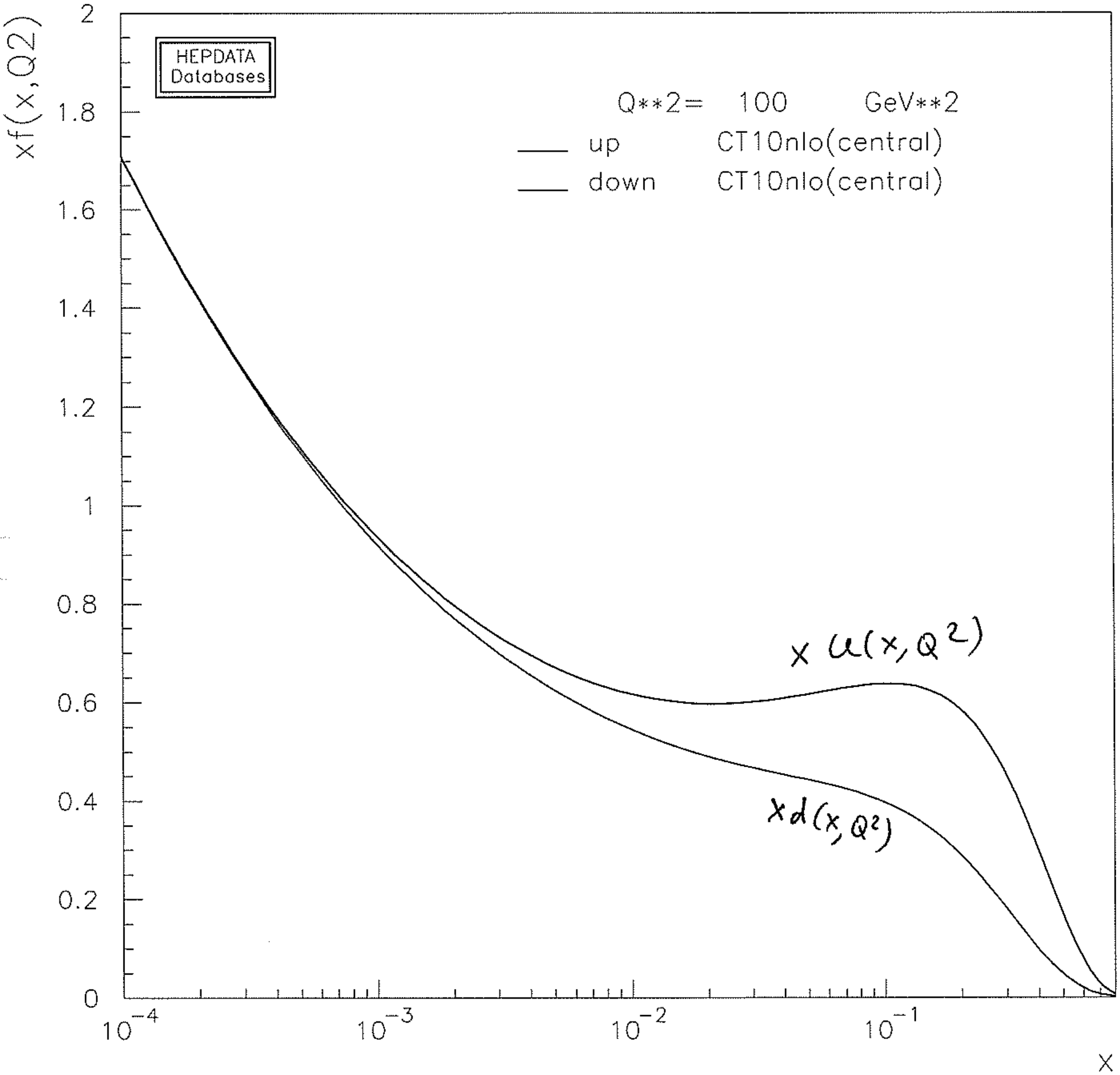
$\Rightarrow$ not so at small-x

outdated cite

go to http://zebu.uoregon.edu/~parton/partongraph.html to plot more.

hepdata.cedar.ac.uk/pdf/pdf3.html ← can plot more if you wish

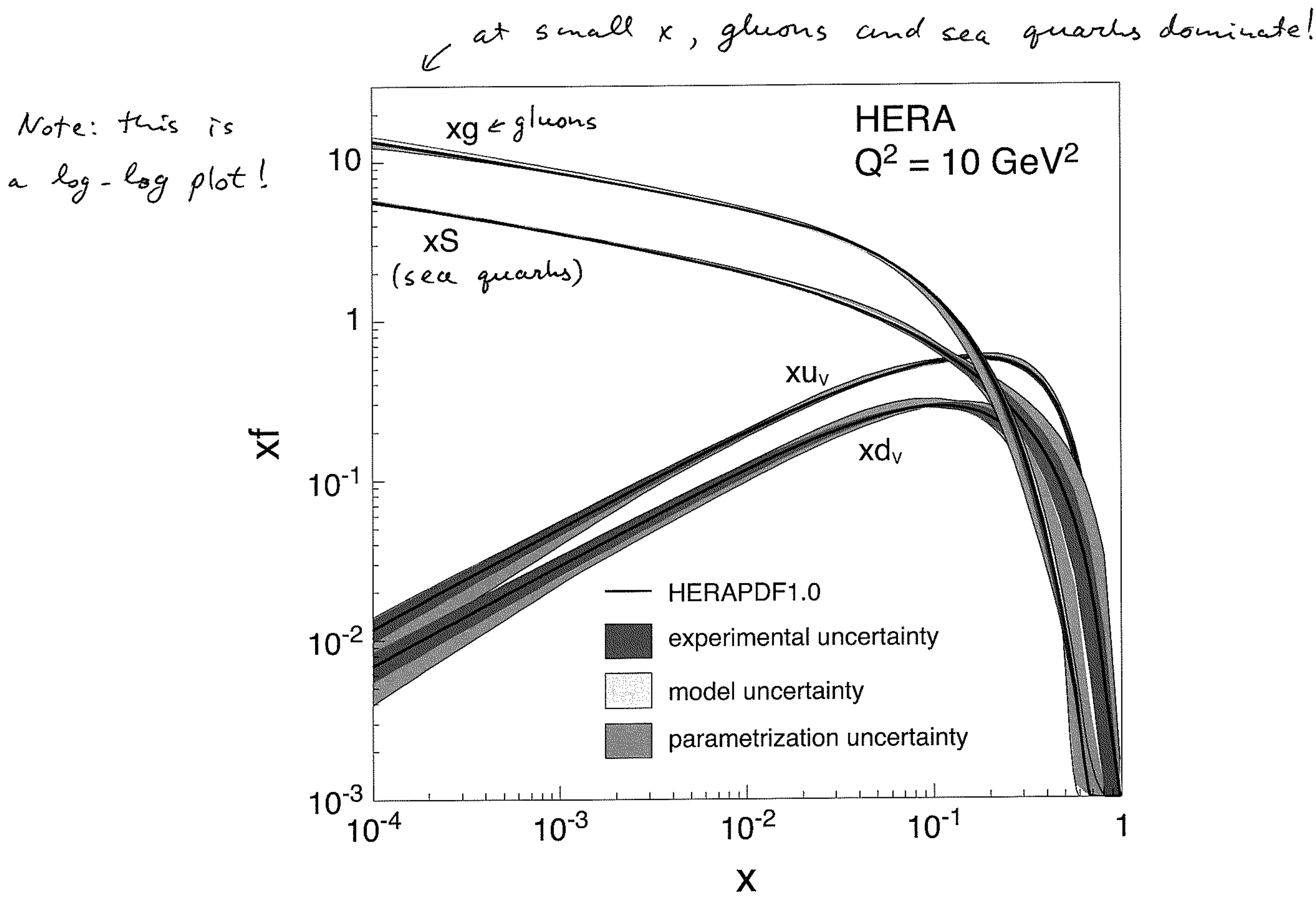
$Q^2 = 100\ GeV^2$ ($\Rightarrow$ $Q = 10\ GeV$)



~ at large x valence quarks dominate

$x u_v \simeq 2\, x d_v$

~ at small x sea quarks dominate


at small x, gluons and sea quarks dominate!
Note: this is a log-log plot!
HERA
Q² = 10 GeV²
xg ← gluons
xS
(sea quarks)
xu_v
xd_v
HERAPDF1.0
experimental uncertainty
model uncertainty
parametrization uncertainty
xf
x
10
1
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻³
10⁻²
10⁻¹
1