

# Homework Set No. 4, Physics 880.02

Deadline – Thursday, June 4, 2009

1. (20 pts) Similar to what we did in class, solve the DGLAP equation for gluon distribution

$$Q^2 \frac{\partial}{\partial Q^2} G(x, Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dx'}{x'} \gamma_{GG}(x/x') G(x', Q^2)$$

with

$$\gamma_{GG}(z) = \frac{2 N_c}{z}$$

in the small- $x$  asymptotics, but now with *fixed* coupling constant  $\alpha_s$  (independent of  $Q^2$ ). In particular show that, in the saddle point approximation, the small- $x$  asymptotics for gluon distribution is given by

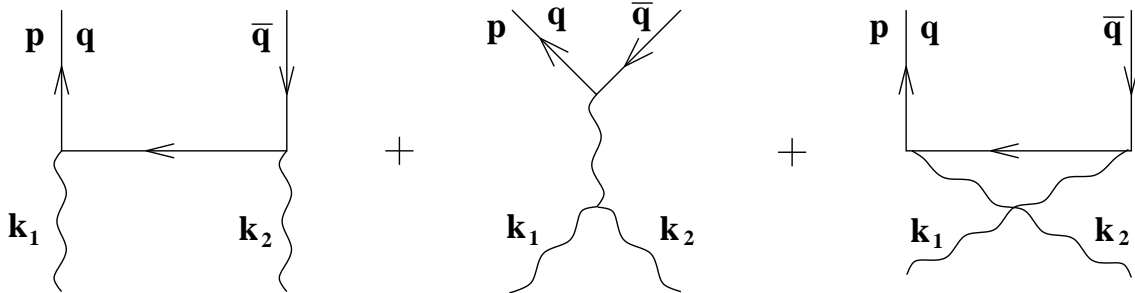
$$xG(x, Q^2) \sim \exp \left( 2 \sqrt{\frac{\alpha_s N_c}{\pi} \ln \frac{1}{x} \ln \frac{Q^2}{Q_0^2}} \right).$$

This is called double logarithmic approximation (DLA) of DGLAP, since we resum both  $\ln \frac{1}{x}$  and  $\ln \frac{Q^2}{Q_0^2}$  in the new parameter  $\alpha_s \ln \frac{1}{x} \ln \frac{Q^2}{Q_0^2}$ .

2. (30 pts) Calculate the cross section for

gluon + gluon  $\rightarrow$  quark + antiquark

at the Born level shown in the figure below. The figure is for the *amplitude*, which needs to be squared and multiplied by appropriate factors to get the cross section.



You should find

$$\mathcal{E}_p \frac{d\hat{\sigma}_{gg \rightarrow q\bar{q}}}{d^3p} = \frac{3\alpha_s^2}{8\hat{s}} (\hat{t}^2 + \hat{u}^2) \left( \frac{4}{9\hat{t}\hat{u}} - \frac{1}{\hat{s}^2} \right) 2\delta(\hat{s} + \hat{t} + \hat{u}) \quad (1)$$

with the Mandelstam variables  $\hat{s} = (k_1 + k_2)^2$ ,  $\hat{t} = (k_1 - p)^2$ ,  $\hat{u} = (k_2 - p)^2$ . The factor of 2 in front of the  $\delta$ -function in Eq. (1) comes from the fact that either the quark or the antiquark can carry momentum  $p$ . ( $q$  and  $\bar{q}$  in the figure denote the quark and the antiquark. Time flows upward.) Assume that quarks are massless.

As a starting point you may take the formula derived in class:

$$\mathcal{E}_p \frac{d\hat{\sigma}_{gg \rightarrow q\bar{q}}}{d^3p} = \frac{1}{16(2\pi)^2 \mathcal{E}_1 \mathcal{E}_2} 2 \delta(\hat{s} + \hat{t} + \hat{u}) \langle |M|^2 \rangle \quad (2)$$

where  $\mathcal{E}_1 = k_1^0$ ,  $\mathcal{E}_2 = k_2^0$ , and  $M$  is the amplitude (sum of all three diagrams drawn above). Angle brackets  $\langle \dots \rangle$  denote summation over final state quantum numbers (polarizations and colors of the quarks) and averaging over initial state quantum numbers (polarizations and colors of the gluons). The factor of 2 mentioned above in Eq. (1) is already included in Eq. (2).

*Hints:*

One quick and dirty way of arriving at the solution is to use the fact that incoming gluons are physical,

$$k^\mu \epsilon_\mu^\lambda(k) = 0, \quad (3)$$

in the amplitude to eliminate some of the terms. Then after squaring the amplitude you may replace the polarization sums by

$$\sum_{\lambda=\pm 1} \epsilon_\mu^{*\lambda}(k) \epsilon_\nu^\lambda(k) \rightarrow -g_{\mu\nu}. \quad (4)$$

Alternatively you may use Eq. (4) without Eq. (3). However, if you follow this path you would have to subtract ghost loop contributions (amplitudes with ghost-antighost pair in initial state instead of gluons, see Serman pp. 233-237). This strategy is a more systematic way of arriving at the right answer.

Other ways of solving the problem include the use of explicit parameterizations for polarization vectors in calculating amplitudes. One may also use polarization sum

$$\sum_{\lambda=\pm 1} \epsilon_\mu^{*\lambda}(k) \epsilon_\nu^\lambda(k) = -g_{\mu\nu} + \frac{\bar{k}_\mu k_\nu + k_\mu \bar{k}_\nu}{k \cdot \bar{k}},$$

where for  $k^\mu = (k^0, \vec{k})$  we defined  $\bar{k}^\mu = (k^0, -\vec{k})$ .

The following  $\gamma$ -matrix formulas may be useful:

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2 \gamma^\nu$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4 g^{\nu\rho}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2 \gamma^\sigma \gamma^\rho \gamma^\nu$$

$$\text{tr}[\gamma^\mu \gamma^\nu] = 4 g^{\mu\nu}$$

$$\text{tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}).$$

For color traces the following expressions may come in handy:

$$T^a T^a = C_F \mathbf{1}$$

with

$$C_F = \frac{N_c^2 - 1}{2 N_c},$$

$$\text{tr}[T^a T^b T^a T^b] = -\frac{C_F}{2}$$

$$f^{abc} f^{abd} = N_c \delta^{cd}.$$