Last time: revisited quark symmetries in QCD:

$N_f = 2$: $\mathcal{L}^{N_f=2}_{\text{quarks}} = \bar{q}\,[\,i\gamma\cdot\partial - m\,]\,q$, $q = \begin{pmatrix} u \\ d \end{pmatrix}$, $m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$

If $m_u = m_d \Rightarrow \mathcal{L}$ is invariant under $SU(2)_{\text{flavor}}$:

$$q \to q' = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\, q, \quad \bar{q} \to \bar{q}' = \bar{q}\, e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}, \quad \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3).$$

$N_f = 3$: $q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, $m = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \Rightarrow \mathcal{L}^{N_f=3}_{\text{quarks}} = \bar{q}\,[\,i\gamma\cdot\partial - m\,]\,q$

$\Rightarrow$ if $m_u = m_d = m_s \Rightarrow \mathcal{L}$ is invariant under $SU(3)_{\text{flavor}}$.

$SU(3)_{\text{flavor}}$: <u>mesons</u> $3 \otimes \bar{3} = 1 \oplus 8 \Rightarrow$ get flavor-octet

$\Rightarrow$ the Eightfold way!

<u>baryons</u>: $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10 \Rightarrow$ get flavor-octet and flavor-decuplet of baryons ~ as is observed!

<u>Flavor SU(2) and SU(3) Symmetries</u> (cont'd)

$N_f = 2$, $\underline{m_u = m_d = 0} \Rightarrow \mathcal{L} = \bar{q}\, i\gamma\cdot\partial\, q \Rightarrow$

define $q = \underbrace{\frac{1-\gamma_5}{2}q}_{q_L} + \underbrace{\frac{1+\gamma_5}{2}q}_{q_R} \Rightarrow \boxed{\mathcal{L} = \bar{q}_L\, i\gamma\cdot\partial\, q_L + \bar{q}_R\, i\gamma\cdot\partial\, q_R}$.

$\Rightarrow$ invariant under $\boxed{SU(2)_L \otimes SU(2)_R}$ chiral symmetry.

put $\underline{m_u = m_d \neq 0}$: $\mathcal{L} = \bar{q}_L\, i\gamma\cdot\partial\, q_L + \bar{q}_R\, i\gamma\cdot\partial\, q_R - m[\bar{q}_L q_R + \bar{q}_R q_L]$

$\Rightarrow SU(2)_L \otimes SU(2)_R$ is broken down to $SU(2)$.

What we know so far: for $N_f = 2$

$$\boxed{SU(2)_L \otimes SU(2)_R} \xrightarrow{m_u = m_d \neq 0} \boxed{SU(2)} \xrightarrow{m_u \neq m_d \neq 0} \text{Nothing}$$

similarly for $N_f = 3$:

$$\boxed{SU(3)_L \otimes SU(3)_R} \xrightarrow{m_u = m_d = m_s \neq 0} \boxed{SU(3)} \xrightarrow{m_u \neq m_d \neq m_s \neq 0} \text{Nothing}$$

$\Rightarrow SU(2)_L \otimes SU(2)_R$ is broken down to $SU(2)$.

What are the conserved currents of $SU(2)_R \otimes SU(2)_L$?

Noether theorem: every symmetry gives a conservation law!

Go back to the massless case:

$$\mathcal{L} = \bar{q}_L i\gamma\cdot\partial\, q_L + \bar{q}_R\, i\gamma\cdot\partial\, q_R$$

$$q_L \underset{SU(2)}{\rightarrow} e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} q_L \Rightarrow \text{if } \vec{\alpha} \text{ is small } q_L \rightarrow \left(1 + i\delta\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}\right) q_L$$
$$= q_L + \delta q_L$$

$\Rightarrow \delta\mathcal{L} = 0$ as it is a symmetry $\Rightarrow$

$$0 = \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta q_L}\delta q_L + \frac{\delta\mathcal{L}}{\delta\bar{q}_L}\delta\bar{q}_L + \frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\delta(\partial_\mu q_L) +$$

$$+ \delta(\partial_\mu\bar{q}_L)\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)} = \underbrace{\left[\frac{\delta\mathcal{L}}{\delta q_L} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\right]}_{=0\ (EOM)}\delta q_L + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\delta q_L\right)$$

$$+ \delta\bar{q}_L\underbrace{\left[\frac{\delta\mathcal{L}}{\delta\bar{q}_L} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)}\right]}_{=0\ (EOM)} + \partial_\mu\left[\delta\bar{q}_L\,\cancel{\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)}}\right]$$

$\rightarrow 0$ for our $\mathcal{L}$

$$\Rightarrow 0 = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\delta q_L\right)$$

$$\Rightarrow 0 = \partial_\mu\left[\bar{q}_L i\gamma^\mu \delta q_L\right] = \partial_\mu\left[\bar{q}_L i\gamma^\mu i\frac{\delta\vec{\alpha}\cdot\vec{\sigma}}{2} q_L\right]$$

$\Rightarrow \partial_\mu j_L^{i\mu} = 0$ where $j_L^{i\mu} = \bar{q}_L \gamma^\mu \frac{\sigma^i}{2} q_L$

left-handed isospin current.

Similarly define $j_R^{i\mu} = \bar{q}_R \gamma^\mu \frac{\sigma^i}{2} q_R$ ~ right-handed isospin current.

$\partial_\mu j_R^{i\mu} = 0$

Alternatively can define

$j_\mu^i = j_{L\mu}^i + j_{R\mu}^i \quad = \bar{q}\, \gamma_\mu \frac{\sigma^i}{2} q$ ~ vector isospin current

$j_{5\mu}^i = j_{R\mu}^i - j_{L\mu}^i = \bar{q}\, \gamma_\mu \gamma_5 \frac{\sigma^i}{2} q$ ~ axial vector isospin current

<u>Define</u> charges: $Q_{L,R}^i(t) = \int d^3x \; j_{L,R\,0}^i(\vec{x}, t)$

$$\frac{dQ_L^i(t)}{dt} = \int d^3x \, \frac{\partial j_{L0}^i(\vec{x},t)}{\partial t} = \int d^3x \left[\underbrace{\partial_\mu j_L^{i\mu}}_{=0 \text{ (conserved current)}} - \vec{\nabla}\cdot \vec{j}_L^{\,i} \right]$$

$$= -\int d^3x \; \vec{\nabla}\cdot\vec{j}_L^{\,i} = 0$$ ← surface term

$\Rightarrow$ charges are conserved!

$\Rightarrow$ the charges are <u>generators</u> of $SU(2)_L \otimes SU(2)_R$!

One can show that they form the chiral $SU(2)_L \otimes SU(2)_R$ algebra:

$[Q_L^i, Q_L^j] = i\, \varepsilon_{ijk} Q_L^k$ ~ $SU(2)_L$

$[Q_R^i, Q_R^j] = i\, \varepsilon_{ijk} Q_R^k$ ~ $SU(2)_R$

$[Q_L^i, Q_R^j] = 0$ ~ commute with each other.

Let's show how Q_L^i generate $SU(2)_L$ transformations.

Let's calculate $[\, Q_L^i(t),\ q_L(t,\vec{x})\,]$:

flavor index 1,2

$$[\, Q_L{}^i(t),\ q_L{}^a_\alpha(\vec{x},t)\,] = \int d^3x' \left[\, \bar{q}_L \gamma_0 \frac{\sigma^i}{2} q_L(\vec{x}',t), \right.$$

spinor index 1,2,3,4

$$\left. q_L{}^a_\alpha(\vec{x},t)\,\right] = \int d^3x' \left[\, \underbrace{\bar{q}_L{}^b_\beta(\vec{x}',t)\,(\gamma^0)_{\beta\delta}}_{q^{+\,b}_{L\,\delta}(\vec{x}',t)} \left(\frac{\sigma^i}{2}\right)_{bc} q_L{}^c_\delta(\vec{x}',t), \right.$$

$$\left. q_L{}^a_\alpha(\vec{x},t)\,\right] =$$

$q^{+\,b}_{L\,\delta}(\vec{x}',t) = q^{+\,b}_{\delta'}\left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta}$, $\quad q_L{}^c_\delta = \left(\frac{1-\gamma_5}{2}\right)_{\delta\delta''} q^c_{\delta''}$

$$= \int d^3x' \cdot \left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta} \left(\frac{1-\gamma_5}{2}\right)_{\delta\delta''} \left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \cdot \left(\frac{\sigma^i}{2}\right)_{bc} \cdot$$

$$\left[\, q^{+b}_{\delta'}(\vec{x}',t)\, q^c_{\delta''}(\vec{x}',t),\ q^a_{\alpha'}(\vec{x},t)\,\right]$$

$\Rightarrow$ use the anti-commutation relations

$$\{\, q^a_\alpha(\vec{x},t),\ q^{+b}_\beta(\vec{x}',t)\,\} = \delta^{ab}\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}').$$

$$[\, Q_L{}^i(t),\ q_L{}^a_\alpha(\vec{x},t)\,] = \left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta''} \left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \left(\frac{\sigma^i}{2}\right)_{bc} \cdot \int d^3x' \cdot$$

$$\cdot(-)\,\{\, q^a_{\alpha'}(\vec{x},t),\ q^{+b}_{\delta'}(\vec{x}',t)\,\}\cdot q^c_{\delta''}(\vec{x}',t) = -\left(\frac{1-\gamma_5}{2}\right)_{\delta'\delta''} \cdot$$

$$\cdot\left(\frac{1-\gamma_5}{2}\right)_{\alpha\alpha'} \left(\frac{\sigma^i}{2}\right)_{bc} \delta^{ab}\,\delta_{\alpha'\delta'}\, q^c_{\delta''}(\vec{x},t) = -\left(\frac{\sigma^i}{2}\right)_{ac} \cdot$$

$$\cdot\left(\frac{1-\gamma_5}{2}\right)_{\alpha\delta''} q^c_{\delta''}(\vec{x},t) = -\left(\frac{\sigma^i}{2}\right)_{ac} q_L{}^c_\alpha(\vec{x},t)$$

$\Rightarrow$ get $\left[Q_L^i(t),\ q_L(\vec{x},t) \right] = -\frac{\sigma^i}{2}\, q_L(\vec{x},t)$

$\Rightarrow$ can show that

$$e^{-i\vec{\alpha}_L\cdot\vec{Q}_L(t)}\, q_L(\vec{x},t)\, e^{i\vec{\alpha}_L\cdot\vec{Q}_L(t)} = e^{i\vec{\alpha}_L\cdot\frac{\vec{\sigma}}{2}}\, q_L(\vec{x},t)$$

$\Rightarrow$ Q_L's generate transformations of $SU(2)_L$

$\Rightarrow$ Q_R's —"— of $SU(2)_R$ (can show similarly).

c.f. $\hat{O}(t) = e^{i\hat{H}t}\,\hat{O}(0)\,e^{-i\hat{H}t} = e^{t\frac{\partial}{\partial t'}}\,\hat{O}(t')\big|_{t'=0}$ $\Rightarrow$ $\hat{H}$ generates time translations

bring back the strange quark $\Rightarrow$ now can perform the same decomposition and for $m_u = m_d = m_s = 0$ have $SU(3)_R \otimes SU(3)_L$ chiral symmetry.

$$\mathcal{L} = \bar{q}_L\, i\gamma\cdot\partial\, q_L + \bar{q}_R\, i\gamma\cdot\partial\, q_R$$

$\Rightarrow$ invariant under $q_L \to e^{i\vec{\alpha}_L\cdot\vec{T}}\, q_L$, $q_R \to e^{i\vec{\alpha}_R\cdot\vec{T}}\, q_R$

$T^a = \frac{\lambda^a}{2}$ ~ generators of $SU(3)$.

Problem: $SU(3)_L \otimes SU(3)_R$ would imply twice as many degenerate multiplets of hadrons: 8 0^- mesons should come in together with 8 0^+ mesons, etc.

$\Rightarrow$ This does not happen in nature. Why?

$\Rightarrow$ you may say: well, as $m_u, m_d, m_s \neq 0$

$\Rightarrow SU(3)_R \otimes SU(3)_L$ is broken.

But then one would not have any multiplets at all, one would not have the Eight fold Way, etc...

$\Rightarrow$ OK, you would say, we have SU(3) flavor if $m_u = m_d = m_s \neq 0$.

$\Rightarrow$ But as $m_u \neq m_d \neq m_s$ SU(3) flavor is just as broken as $SU(3)_L \otimes SU(3)_R$.

(NB) Fact of the matter is both $SU(3)_L \otimes SU(3)_R$ and SU(3) are broken "slightly". The real symmetry breaking is $SU(3)_L \otimes SU(3)_R \to SU(3)$ is done through <u>spontaneous symmetry breaking</u>. (SSB)

SSB has <u>nothing</u> to do with quark masses!

$\Rightarrow$ SSB would also explain why the masses of hadrons are so much higher than the current masses of quarks they are made of.

$(m_p = 938 \text{ MeV}, \quad 2m_u + m_d < 30 \text{ MeV})$

$$\frac{2m_u + m_d}{m_p} \simeq 3\%.$$

$$SU(3)_L \otimes SU(3)_R$$

$$\Big\downarrow \; m_u = m_d = m_s \neq 0$$

$$SU(3)$$

$$\Big\downarrow \; m_u \neq m_d \neq m_s \neq 0$$

Nothing

equally broken

$\Rightarrow$ $SU(3)$ works (the Eight fold way) approximately

$\Rightarrow$ $SU(3)_L \otimes SU(3)_R$ <u>does not work!</u> (no 0^+ meson octet, $\left(\frac{1}{2}\right)^-$ baryon octet, ...)

Solution:

$$SU(3)_L \otimes SU(3)_R$$

$$\Big\downarrow \; m_u = m_d = m_s \neq 0 \;\; \oplus \;\; \underline{\underline{SSB}}$$

(Spontaneous (chiral) Symmetry Breaking)

SSB makes this breaking much worse than that

$$SU(3)$$

$$\Big\downarrow \; m_u \neq m_d \neq m_s \neq 0$$

Nothing

Spontaneous (Chiral) Symmetry Breaking.

General Discussion: Spontaneous Symmetry Breaking.

Def. Spontaneous Symmetry Breaking (SSB): a symmetry which is manifest in Lagrangian (and Hamiltonian), but is not respected by the ground state of the system.

Example: Ising model in 2d: $H = -\sum_{\substack{\text{nearest}\\ \text{neighbours}\\ i,j}} \frac{J}{4} s_i s_j$

$J > 0 \Rightarrow$ spins tend to align

$s_i = \pm 1$ ~ projection of spins on y-axis

$\Rightarrow$ the system is up-down symmetric: H is invariant under $s_i \to -s_i$.

$\Rightarrow$ However, the system spontaneously choses a ground state, which is either all spins up or all spins down:

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \end{array} \quad \text{or} \quad \begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

In this ground state one has $\langle S_i \rangle \neq 0$

~ non-zero magnetization $\Rightarrow$ $S_i \to -S_i$ invariance is lost.

(Note that $S_i \to -S_i$ is still a symmetry of H!)

<u>Landau - Ginzburg theory of ferromagnetism</u>:

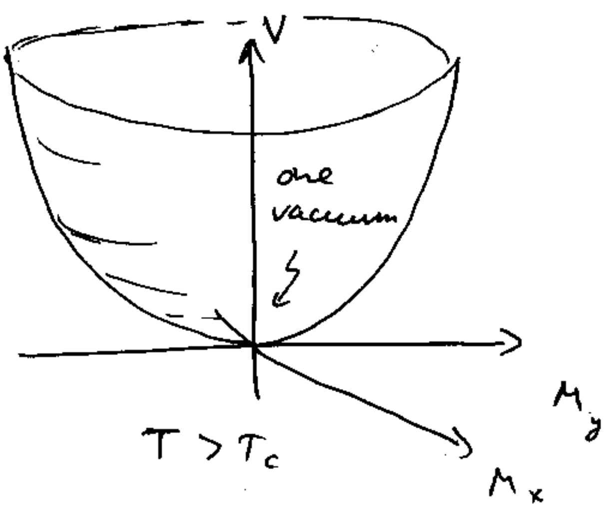
$$H = \int d^3x \left[\left(\nabla_i \cdot M_j \right)^2 + \underbrace{\mu^2 (T - T_c) \, \vec{M}^2 + \lambda \left(\vec{M}^2 \right)^2}_{V(\vec{M}) \,\sim\, \text{the potential.}} \right]$$

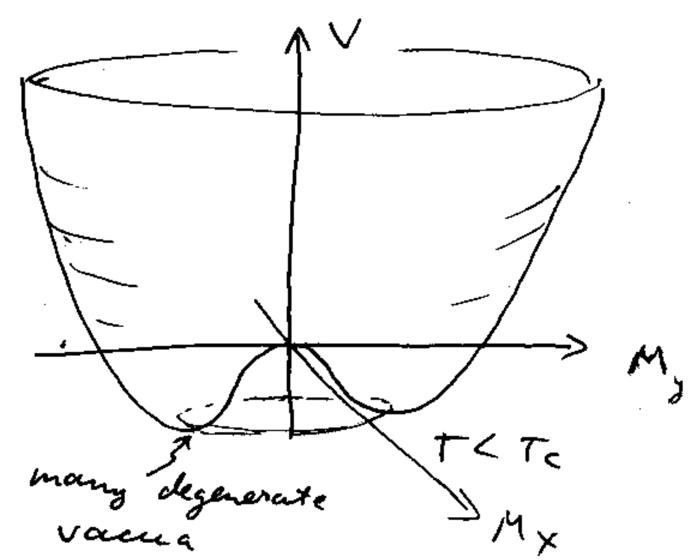
$\vec{M} = (M_1, M_2, M_3)$ is magnetization of the medium

$\lambda > 0$ ~ constant, $\mu^2 > 0$ ~ constant

T ~ temperature, T_c ~ critical (Curie) temperature.

Let's plot the potential $V(\vec{M})$: (assume 2d system)





"Mexican hat" potential

$\Rightarrow$ the Hamiltonian is symmetric under spatial rotations: $M^i \to M'^i = R^{ij} M^j$

$x'_i = R_{ij} x_j \Rightarrow |\vec{x}'|^2 = |\vec{x}|^2 \Rightarrow R_{ij} x_j R_{ik} x_k = x_i x_i$

$\Rightarrow R_{ij} R_{ik} = \delta_{jk} \Rightarrow R \cdot R^T = R^T R = \mathbb{1} \Rightarrow$ forget reflections $\Rightarrow$ require $\det R = +1 \Rightarrow SO(3)$

~ a group of special ($\det = +1$) real orthogonal ($RR^T = R^T R = \mathbb{1}$) 3×3 matrices.

$\Rightarrow$ for $T < T_c$ the ground state is at the minima

$$\Rightarrow \quad \mu^2 (T - T_c)\, 2 |\vec{M}| + 4\lambda |\vec{M}|^3 = 0$$

$$\Rightarrow \quad |\vec{M}_{vac}| = \sqrt{\frac{\mu^2 (T_c - T)}{2\lambda}}$$

$\Rightarrow$ however, direction of $\vec{M}$ is chosen spontaneously!

Say, $M_{vac} = \sqrt{\frac{\mu^2 (T_c - T)}{2\lambda}}\ \hat{x} \equiv |0\rangle$

define generators of $SO(3)$: $L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$, $L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$,

$L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow e^{-i \vec{\alpha} \cdot \vec{L}}$ is a rotation by angle $|\vec{\alpha}|$ around $\vec{\alpha}$-direction.

$\Rightarrow H$ is invariant under $\vec{M} \to \vec{M}' = e^{-i\vec{\alpha} \cdot \vec{L}} \vec{M}$.

$\Rightarrow$ ground state is not rotationally symmetric:

$R|0\rangle \neq |0\rangle \Rightarrow$ if $R = e^{i \vec{\alpha} \cdot \vec{Q}}$, $\vec{Q}$ ~ conserved charges of symmetry $\Rightarrow Q^i |0\rangle \neq 0$ $\left(\begin{matrix}\text{equivalently} \\ \langle 0 | \vec{M} | 0 \rangle \neq 0\end{matrix}\right)$