Last time: finished talking about SSB

~ talked about π^+, π^-, π^0 as Goldstone bosons of

S chiral SB : $SU(2)_L \otimes SU(2)_R \to SU(2)$

~ $SU(3)_L \otimes SU(3)_R \to SU(3)$ for $N_f = 3 \Rightarrow$ get 8

Goldstone bosons: $\pi^+, \pi^-, \pi^0, K^+, K^0, \bar{K}^0, K^-, \eta^0$

~ VEV for this SSB is $\langle 0 | \bar{q} q | 0 \rangle \simeq -(230\,\text{MeV})^3$.
chiral (quark) condensate.

The Electroweak Theory (cont'd)

Local Gauge Symmetries (cont'd)

$U(1)$ global: $\mathcal{L} = \bar{\psi}[i\gamma \cdot \partial - m]\psi$ QED electrons

to make $U(1)$ symmetry local has to gauge the $\mathcal{L}$:

introduce a gauge field $A_\mu(x)$:

$$\mathcal{L}_{QED} = \bar{\psi}[i\gamma^\mu D_\mu - m]\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad D_\mu = \partial_\mu - ig A_\mu$$

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$\Rightarrow \mathcal{L}_{QED}$ is symmetric under local $U(1)$ symmetry:

$$\begin{cases} \psi \to e^{i\alpha(x)}\psi \\ A_\mu \to A_\mu + \frac{1}{g}\partial_\mu \alpha(x) \end{cases} \quad \text{gauge transformation}$$

$\Rightarrow$ what about non-Abelian groups?

$\Rightarrow$ take $SU(2)$: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \Rightarrow \mathcal{L} = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m\right]\psi$

with $m = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ has a <u>global</u> $SU(2)$ symmetry.

$\Rightarrow$ Make it <u>local</u> $\Rightarrow$ gauge the Lagrangian by introducing a gauge field $A_\mu^a(x)$, $a = 1, 2, 3$ (three gauge fields) $\Rightarrow$

$$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}, \quad D_\mu = \partial_\mu - ig A_\mu$$

(summation over a's, μ's, ν's is assumed)

$$A_\mu = A_\mu^a \frac{\tau^a}{2} = A_\mu^1 \frac{\tau^1}{2} + A_\mu^2 \frac{\tau^2}{2} + A_\mu^3 \frac{\tau^3}{2}$$

$\Rightarrow$ we saw that for $\bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi$ to be gauge-inv.

need
$$\begin{cases} \psi' = S(x)\psi \\ A_\mu' = S(x) A_\mu S^{-1}(x) - \frac{i}{g}\left(\partial_\mu S\right) S^{-1} \end{cases}$$
non-Abelian gauge transformation

$$S(x) = e^{i\vec{\alpha}(x)\cdot\frac{\vec{\tau}}{2}}.$$

$\Rightarrow$ but what is $F_{\mu\nu}$? $\quad F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right]$ like in abelian case

$$F_{\mu\nu} = F_{\mu\nu}^a \frac{\tau^a}{2}.$$

$\Rightarrow$ is $F_{\mu\nu}^a F^{a\mu\nu}$ gauge invariant?

where $F_{\mu\nu} = F_{\mu\nu}^{a} \frac{\tau^a}{2}$.

$$F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right] = \frac{i}{g}\left[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu\right] =$$

$$= \frac{i}{g}\left\{-ig\left[\partial_\mu, A_\nu\right] - ig\left[A_\mu, \partial_\nu\right] - g^2\left[A_\mu, A_\nu\right]\right\}$$

$$= \frac{i}{g}\left\{-ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - g^2\left[A_\mu, A_\nu\right]\right\} =$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$$

$$\Rightarrow \boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]}$$

$$F_{\mu\nu}^{a}\frac{\tau^a}{2} = \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\frac{\tau^a}{2} - ig A_\mu^b A_\nu^c \underbrace{\left[\frac{\tau^b}{2}, \frac{\tau^c}{2}\right]}_{i\varepsilon^{bca}\frac{\tau^a}{2}} \leftarrow SU(2)$$

$$= \frac{\tau^a}{2}\left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,\varepsilon^{abc} A_\mu^b A_\nu^c\right]$$

$$\Rightarrow \boxed{F_{\mu\nu}^{a} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,\varepsilon^{abc} A_\mu^b A_\nu^c}$$

~ true for SU(2)

~ other groups have different group structure constants:

$$F_{\mu\nu}^{a} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c.$$

What happens to $F_{\mu\nu}$ under non-Abelian gauge transform?

Start with D_μ: $D_\mu = \partial_\mu - ig A_\mu \to$

$$\to \partial_\mu - ig\left[S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S) S^{-1}\right] =$$

$$= S\left[\partial_\mu - ig A_\mu\right] S^{-1} = S D_\mu S^{-1}$$

as $S \partial_\mu S^{-1} = \partial_\mu + S(\partial_\mu S^{-1})$

now: $\mathbb{1} = S S^{-1} \Rightarrow 0 = \partial_\mu (S S^{-1}) = (\partial_\mu S) S^{-1} + S(\partial_\mu S^{-1})$

$\Rightarrow S(\partial_\mu S^{-1}) = -(\partial_\mu S) S^{-1} \Rightarrow S \partial_\mu S^{-1} = \partial_\mu - (\partial_\mu S) S^{-1}$

$$\Rightarrow \boxed{D_\mu \to S D_\mu S^{-1}}$$

$$\Rightarrow F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] \to \frac{i}{g}[S D_\mu S^{-1}, S D_\nu S^{-1}]$$

$$= \frac{i}{g} S [D_\mu, D_\nu] S^{-1} = S F_{\mu\nu} S^{-1}$$

$$\Rightarrow \boxed{F_{\mu\nu} \to F'_{\mu\nu} = S F_{\mu\nu} S^{-1}}$$

$\Rightarrow$ Note that $F_{\mu\nu}$ is <u>not</u> invariant under gauge transformation if it is non-Abelian!

$$-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} \, tr\left(F_{\mu\nu} F^{\mu\nu}\right)$$

as $tr\left(\frac{\tau^a}{2}\frac{\tau^b}{2}\right) = \frac{1}{2}\delta^{ab}$ => under non-abelian gauge transformation have

$$-\frac{1}{2} tr\left(F_{\mu\nu} F^{\mu\nu}\right) \to -\frac{1}{2} tr\left(F'_{\mu\nu} F'^{\mu\nu}\right) = -\frac{1}{2} tr\left[\cancel{S} F_{\mu\nu} \cancel{S^{-1}} \cancel{S} F^{\mu\nu} \cancel{S^{-1}}\right] = -\frac{1}{2} tr\left[F_{\mu\nu} F^{\mu\nu}\right]$$

=> the Lagrangian is invariant under non-Abelian gauge transformation:

$$\boxed{\mathcal{L} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{2} tr\left(F_{\mu\nu} F^{\mu\nu}\right)}$$ true for any gauge group $SU(N)$

$$D_\mu = \partial_\mu - i g A_\mu \quad , \quad F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right].$$

The Higgs Mechanism ($U(1)$ model)

~ Imagine a case when gauge symmetry is spontaneously broken

~ Goldstone Th'm does not apply: needs manifest Lorentz invariance (VEV is $\mathcal{L}$.inv) & positivity of the norm. (to have G. boson state with $\langle n|n\rangle > 0$)

In gauge theories $\mathcal{L}$. inv. gauges $\partial_\mu A^\mu = 0$ don't have >0 of the norm, other gauges $A^0 = 0$, $\vec{\nabla}\cdot\vec{A} = 0$ are not manifestly $\mathcal{L}$.-inv..

Consider a Lagrangian:

$$\mathcal{L} = (D_\mu \varphi)^* (D_\mu \varphi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mu^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2$$

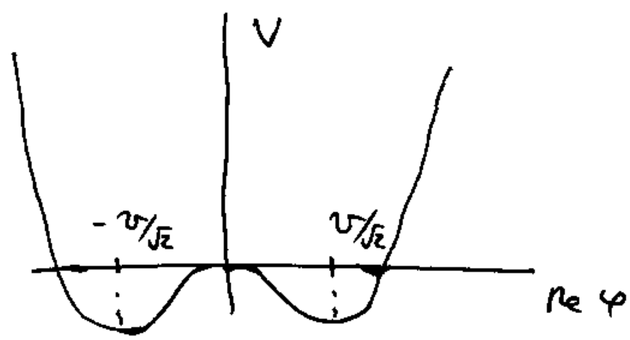
φ ~ complex scalar field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

A_μ ~ abelian gauge field, $D_\mu = \partial_\mu - i g A_\mu$.

The theory has a $U(1)$ local gauge symmetry:

$$\begin{cases} \varphi \to \varphi' = e^{i\alpha(x)} \varphi \\ A_\mu \to A'_\mu = A_\mu + \frac{1}{g} \partial_\mu \alpha \end{cases}$$

The potential is $V(\varphi) = -\mu^2 \varphi^* \varphi + \lambda (\varphi^* \varphi)^2$.



=> the minimum is at

$$-2\mu^2 v' + 4 \lambda v'^3 = 0$$

$$\Rightarrow v' = \frac{\mu}{\sqrt{2\lambda}} \equiv \frac{v}{\sqrt{2}}$$

=> have an ∞ of vacua: $\langle 0 | \varphi | 0 \rangle = v' e^{i\theta(x)}$, θ ~ real

=> pick $\langle 0 | \varphi | 0 \rangle = \frac{v}{\sqrt{2}} = v'$ as the vacuum.

=> SSB of gauge ~~the~~ symmetry!

Write $\varphi = \frac{\rho'(x)}{\sqrt{2}} e^{i\theta(x)}$ with $\rho'(x), \theta(x)$ real fields.

$$\mathcal{L} = \left[\left(\partial_\mu + igA_\mu\right)\frac{\rho'}{\sqrt{2}} e^{-i\theta}\right]\cdot\left[\left(\partial_\mu - igA_\mu\right)\frac{\rho'}{\sqrt{2}} e^{i\theta}\right] -$$

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu^2}{2}(\rho')^2 - \frac{\lambda}{4}(\rho')^4 = \left[\left(\partial_\mu + igA_\mu - i\partial_\mu\theta\right)\frac{\rho'}{\sqrt{2}}\right]$$

$$\cdot\left[\left(\partial_\mu - igA_\mu + i\partial_\mu\theta\right)\frac{\rho'}{\sqrt{2}}\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu^2}{2}(\rho')^2 - \frac{\lambda}{4}(\rho')^4$$

$\Rightarrow$ define $B_\mu \equiv A_\mu - \frac{1}{g}\partial_\mu\theta$ ($F_{\mu\nu}$ remains the same ~ like a gauge transf.)

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial_\mu + igB_\mu\right)\rho'\right]\left[\left(\partial_\mu - igB_\mu\right)\rho'\right] - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{\mu^2}{2}(\rho')^2 - \frac{\lambda}{4}(\rho')^4.$$

with $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = F_{\mu\nu}$

Now, in the new vacuum $\langle 0|\rho'|0\rangle = v$, $\langle 0|\theta|0\rangle = 0$

$\Rightarrow$ write $\rho' = \rho + v$ $\Rightarrow$ get

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial_\mu + igB_\mu\right)(\rho+v)\right]\left[\left(\partial_\mu - igB_\mu\right)(\rho+v)\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu^2}{2}(\rho+v)^2 - \frac{\lambda}{4}(\rho+v)^4.$$

kinetic term $= \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \frac{1}{2}g^2 B_\mu B^\mu(\rho+v)^2 +$

$$+\frac{1}{2}\cancel{\left(igB_\mu(\rho+v)\partial_\mu\rho - igB_\mu(\rho+v)\partial_\mu\rho\right)}^{\,0} = \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho$$

$$+\frac{1}{2}g^2 B_\mu B^\mu(\rho+v)^2.$$

potential: $\frac{\mu^2}{2}\left(2\rho\cdot v+\rho^2\right)-\frac{\lambda}{4}\left(\rho^4+4\rho^3 v+6\rho^2 v^2\right.$
$\left.+4\rho v^3+v^4\right)=\rho\,\cancel{\left(\mu^2 v-\lambda v^3\right)}\to 0+\rho^2\cdot\left(\frac{\mu^2}{2}-\right.$
$\left.-\frac{3}{2}\underbrace{\lambda v^2}_{\mu^2}\right)-\lambda\rho^3 v-\frac{\lambda}{4}\rho^4=-\mu^2\rho^2-\lambda\rho^3 v-\frac{\lambda}{4}\rho^4$

$$\mathcal{L}=\frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho-\mu^2\rho^2-\frac{1}{4}G_{\mu\nu}G^{\mu\nu}+\frac{1}{2}g^2v^2B_\mu B^\mu$$
$$+\frac{1}{2}g^2B_\mu B^\mu\left(2\rho v+\rho^2\right)-\lambda\rho^3 v-\frac{\lambda}{4}\rho^4$$

particle content:

~ a scalar ρ with mass $\mu\sqrt{2}$.

~ a massive gauge field B_μ with mass

$m_B = g v$ (see HW #2, problem on Proca Lagrangian: $\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{m^2}{2}A_\mu A^\mu$.)

~ field θ got "eaten up" by B_μ, as massless gauge field A_μ had 2 d.o.f., now $\oplus$ 1 (θ)

$\Rightarrow$ B_μ has 3 degrees of freedom.

θ ~ "would-be" Goldstone boson

~ if we had not absorbed θ into B_μ would have gotten terms like $A_\mu\partial^\mu\theta$ ~ not clear how to interpret. (related to negative normproblem) (~~———~~ - -) ?