Last time: we defined Lie groups: groups with elements labeled by a continuous parameter $\vec{\alpha}$.

$$D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{X}}$$, where $\vec{X}$ are group generators.

$[X_a, X_b] = i f_{abc} X_c$ is the Lie algebra of generators.

f_{abc}: structure constants

SU(2): a Lie group, $\vec{J} = \frac{\vec{\sigma}}{2}$ are generators,

$[J_i, J_j] = i\varepsilon_{ijk} J_k \Rightarrow \varepsilon_{ijk}$ are SU(2) structure constants

SU(3): $T^a = \frac{\lambda^a}{2}$ are generators, $\lambda^a \sim$ Gell-Mann matrices

$[T^a, T^b] = i f^{abc} T^c \Rightarrow f^{abc}$ are SU(3) structure constants $\sim$ just #'s $\in \mathbb{R}$

$\Rightarrow$ Jacobi identity $[X_a,[X_b,X_c]]$ + permutations $= 0$

$\Rightarrow (t^a)_{bc} = -i f_{abc} \sim$ adjoint representation

Tensor Methods for SU(N)

U-matrices act on N-dim vectors a^i: $a^i \to a'^i = U^i{}_j a^j$

$\Rightarrow \delta^i_j$, $\varepsilon^{i_1 \ldots i_N}$ are invariant under U-actions

$$\varepsilon'_{i_1\ldots i_N} = U_{i_1}{}^{j_1} \ldots U_{i_N}{}^{j_N} \varepsilon_{j_1\ldots j_N} = (\det U)\, \varepsilon_{i_1\ldots i_N} = \varepsilon_{i_1\ldots i_N}$$

$\Rightarrow \delta$'s, ε's raise & lower indices: $\varepsilon_{ijk} a^j b^k = c_i$

a^i ~ give a basis for the space U's act on $\Rightarrow$

$\Rightarrow$ correspond to u's in the fundamental repr. $\mathbf{N}$.

a_i ~ —"— $\bar{\mathbf{N}}$ (conjugate representation)

$\Rightarrow$ to reduce to irreducible representations notice that $P_{12}\, a'^{ij} = U^i{}_k U^j{}_l P_{12}\, a^{kl}$ $\Rightarrow$ (anti-) symmetric tensors transform into (anti-) symmetric ones

$\Rightarrow$ all reductions you can do $\Rightarrow$

<u>Example</u> $$a^i b^j = \frac{1}{2}(a^i b^j + a^j b^i) + \frac{1}{2}\underbrace{(a^i b^j - a^j b^i)}_{\varepsilon^{ijk}\varepsilon_{klm} a^l b^m}$$

$$\Rightarrow \boxed{3 \otimes 3 = 6 \oplus \bar{3}}$$

a^{ij} : $S^{ij} = \frac{1}{2}(a^{ij} + a^{ji})$, $A^{ij} = \frac{1}{2}(a^{ij} - a^{ji})$

$\Rightarrow P_{12} S^{ij} = S^{ij}$; $P_{12} A^{ij} = -A^{ij}$

What is this good for?

Take a product of two representations:

$$a^i b^j = \frac{1}{2}(a^i b^j + a^j b^i) + \frac{1}{2}(a^i b^j - a^j b^i)$$

take $SU(3)$ for example: a^i is $\mathbf{3}$, $a^i b^j$ is $3 \otimes 3$.

$\frac{1}{2}(a^i b^j + a^j b^i)$ has 6 indep. components $\Rightarrow$ ~~transfor~~ makes a basis for representation $\mathbf{6}$.

$\frac{1}{2}(a^i b^j - a^j b^i)$ has 3 indep. components

$$= \frac{1}{2} \varepsilon^{ijk} \underbrace{\varepsilon_{klm} a^l b^m}_{c_k} \Rightarrow \text{it is } \bar{3}$$

$\Rightarrow$ we showed that $3 \otimes 3 = 6 \oplus \bar{3}$.

$$a^i b_j = \underbrace{\left(a^i b_j - \frac{1}{3} \delta^i_j a^k b_k\right)}_{\text{traceless } 3\times 3 \text{ matrix}} + \underbrace{\frac{1}{3} \delta^i_j a^k b_k}_{1 \text{ (a singlet)}}$$

$\Rightarrow$ 8 d.of.freedom $\Rightarrow$ an 8 (adjoint representation)

$\Rightarrow$ $3 \otimes \bar{3} = 8 \oplus 1$.

$\Rightarrow$ for a^i_j one can decompose the result into traceless & dimension 1 subspaces.

<u>Another example</u> $3 \otimes 3 \otimes 3 = ?$

$$a^i b^j c^k = a^i \left[\frac{1}{2}\left(b^j c^k + b^k c^j\right) + \frac{1}{2}\left(b^j c^k - b^k c^j\right) \right] =$$

= lots of algebra ...

<u>Young Tableaux</u>

~ a method to avoid tedious symmetrization, etc.

S^{ij} (symmetric tensor) $\rightarrow$ $\boxed{i}\boxed{j}$

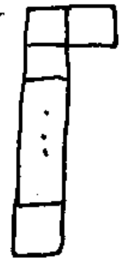
$a^i \rightarrow \boxed{i}$ ~ representation N for $SU(N)$

$$a_{i_1} = \underbrace{\varepsilon_{i_1 i_2 \ldots i_N}}_{\text{Levi-Civita symbol}} b^{i_2 \ldots i_N} \Rightarrow b^{i_2 \ldots i_N} \sim \text{anti-symmetric}$$

$$A^{ij} = \begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array} \quad \text{(anti-symmetrize vertically)}$$

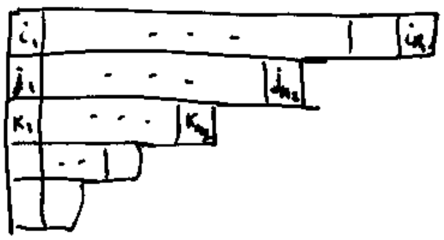
$$\Rightarrow a_i = \left.\begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline\end{array}\right\} N-1 \text{ boxes}, \quad \bar{N} \text{ representation for } SU(N)$$

For $SU(3)$ $\quad a^i b_j - \frac{1}{3}\delta^i_j a^k b_k$ was adjoint $\Rightarrow$

$\Rightarrow$ take $a^i b_j \Rightarrow$  . ~ adjoint Young tableaux

$N-1$ boxes

$\Rightarrow$ general rule: symmetrize in the indices in the rows; then anti-symmetrize in the indices in the columns.



$n_1 \geq n_2 \geq n_3 \ldots$

Example: SU(2): 2 is $a^i \Rightarrow \square$

$\bar{2}$ is $a_i \Rightarrow \square$ too $(N-1=2)$

$\Rightarrow$ 2 and $\bar{2}$ are equivalent.

adjoint representation: $\square\square \sim a^i b_j - \frac{1}{3}\delta^i_j a^k b_k$

$\Rightarrow$ has dimension 3 $\Rightarrow$ triplet

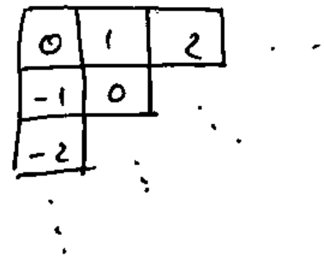
SU(3): 3 is $a^i \Rightarrow \square$, $\bar{3}$ is $a_i \Rightarrow \begin{smallmatrix}\square\\\square\end{smallmatrix}$.

adjoint representation 8 is $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$. (octet).

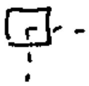
The dimension of a SU(N) representation given by Young tableaux is:

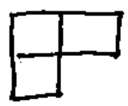
$$d = \prod_{\substack{\text{boxes}\\ i}} (N + D_i)/h_i$$

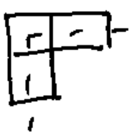
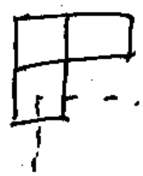
h_i ~ hook length  $h_i = 2$,  $h_i = 1$

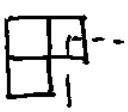
D_i ~ distance to 1st box 

Examples:  in $SU(N)$

 $h_i = 1$ $\boxed{0}$ $\Rightarrow D_i = 0 \Rightarrow d = N$ ~ true!

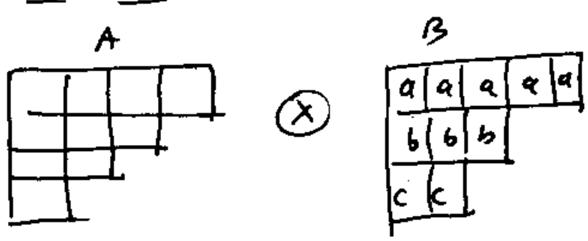
$SU(3)$:  $N = 3$ for $SU(3)$ 

 $h_1 = 3$, $D_1 = 0$  $h_2 = 1$, $D_2 = -1$

 $h_3 = 1$, $D_3 = 1$

$$\Rightarrow d = \prod_{i=1}^{3} \frac{\overset{=3}{N} + D_i}{h_i} = \frac{3}{3} \cdot \frac{2}{1} \cdot \frac{4}{1} = 8$$ ~ works!

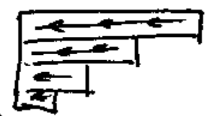
Reduction of Product-Representations:



(i) take boxes a and put them on the tableaux A building right & down (# boxes in rows should always not increase as you go down & # boxes in columns does not increase as you go right), no two a's in the same column

(ii) ibid for b's, c's

(iii) to avoid doublecounting, reading from right to left in any row from top down one must have # a's $\geq$ # b's $\geq$ # c's $\geq$ ...

starting from $\forall$ box in each row:  ~ read this way

$\Downarrow$

get a a b a b c ...

$\Rightarrow$ should be # a's $\geq$ # b's $\geq$ # c's ... to the left from $\forall$ symbol

<u>Examples</u>! SU(3): $\square \otimes \boxed{a} = \boxed{\ }\boxed{a} \oplus \begin{array}{|c|}\hline \ \\ \hline a \\ \hline\end{array}$

$3 \otimes 3 = \ ? \oplus \bar{3}$

$\boxed{0}\boxed{1} \Rightarrow$ $h_1 = 2$, $D_1 = 0$ $\quad$ $h_2 = 1$, $D_2 = 1$

$d = \frac{3}{2} \cdot \frac{4}{1} = 6 \Rightarrow$ this is 6! $\Rightarrow$ $3 \otimes 3 = 6 \oplus \bar{3}$

as we saw before!

What about $3 \otimes \bar{3}$? $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \otimes \boxed{a} = \begin{array}{|c|c|}\hline \ & a \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline a \\ \hline\end{array}$

$\bar{3} \otimes 3 = 8 \oplus \ ?$

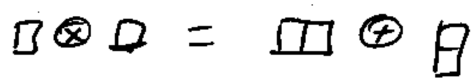
$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$ is $\mathbb{1}$ for SU(3) $\Rightarrow$ a_{ijk} ~ has only one non-trivial component! ~ ε_{ijk}

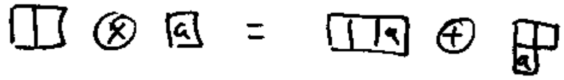
$\Rightarrow$ $\bar{3} \otimes 3 = \mathbb{1} \oplus 8$ as before.

$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} = 0$ for SU(3) ~ can't have more than 3 boxes in a column $\quad$ $a_{ijkl} = 0$ if require it to be anti-symmetric.

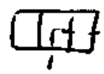
$\Rightarrow$ get a ~~line~~ column of length N for SU(N) ~ a singlet (discard it if it's a part of a larger tableaux)

We wanted to find $3 \otimes 3 \otimes 3$. We know that

$3 \otimes 3 = 6 \oplus \bar{3} \iff$ 

$6 \otimes 3 =$ 

? 8

 $h_1 = 3,\ D_1 = 0$  $h_2 = 2,\ D_2 = 1,$  $h_3 = 1,\ D_3 = 2$

$d = \frac{3}{3} \cdot \frac{4}{2} \cdot \frac{5}{1} = 10 \Rightarrow$  is **10**.

$$6 \otimes 3 = 10 \oplus 8$$

$3 \otimes \bar{3} = 1 \oplus 8$ (know from before)

$$\Rightarrow \boxed{3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10}$$

check: $3^3 = 27$ $1 + 8 + 8 + 10 = 27$ too!

$\Rightarrow$ to learn more about groups read

"Lie Algebras in Particle Physics"

by H. Georgi

$\Rightarrow$ may want to learn about weights & roots..