# Homework Set No. 2, Physics 880.08

# Deadline – Wednesday, April 28, 2010

**1.** (15 pts) Imagine that the full non-perturbative beta-function of QED were

$$\beta(\alpha) \,=\, \frac{\alpha^2}{3\,\pi}\,\left[1 - e^{1-\frac{1}{\alpha}}\right].$$

Find the running QED coupling constant $\alpha(Q^2)$ for such beta-function. Sketch $\alpha(Q^2)$ as a function of $Q^2$. Find the UV fixed point and determine the large-$Q^2$ asymptotics of $\alpha(Q^2)$, i.e., find how it approaches the fixed point.

**2. a.** (15 pts) Consider a harmonic oscillator in a background of a time-dependent external force (source) $j(t)$. The Lagrangian is

$$L \,=\, \frac{1}{2}\,m\,\dot{q}^2 - \frac{1}{2}\,m\,\omega^2\,q^2 + q\,j(t).$$

Using quasi-classical method for evaluation of path integrals find the time-evolution (Feynman) kernel

$$\begin{aligned} U(q_f, t_f; q_i, t_i) \,&=\, {}_S\langle q_f(t_f)|\, e^{-\frac{i}{\hbar}\,\hat{H}\,(t_f - t_i)}\,|q_i(t_i)\rangle_S \,=\, {}_H\langle q_f, t_f | q_i, t_i\rangle_H \\ &=\, \int [\mathcal{D}q]\,\exp\left\{\frac{i}{\hbar}\int\limits_{t_i}^{t_f} dt\,L(t)\right\}. \end{aligned}$$

You may use the result derived in class for the harmonic oscillator without the external force, though this time you also need to find the classical action in terms of $j(t)$. In evaluating the classical action assume that $q_i = q_f = 0$, or, more specifically, require that $q_{cl}(t) = 0$ when $j(t) = 0$. When solving classical EOM you may find Fourier-integral decomposition

$$q(t) \,=\, \int\limits_{-\infty}^{\infty} \frac{dE}{2\,\pi}\,e^{-\frac{i}{\hbar}\,E\,t}\,q_E$$

useful.

**b.** (10 pts) Use the result of part **a** to show that the two-point function for the harmonic oscillator without the external force is given by

$$\langle 0|\,\mathrm{T}\,\hat{q}(t_1)\,\hat{q}(t_2)\,|0\rangle \,=\, i\,\frac{\hbar^2}{m}\int\limits_{-\infty}^{\infty}\frac{dE}{2\,\pi}\,\frac{e^{-\frac{i}{\hbar}\,E\,(t_1 - t_2)}}{E^2 - \hbar^2\,\omega^2 + i\,\epsilon}. \tag{1}$$

**c.** (10 pts) Re-derive the two-point function in Eq. (1) by using creation and annihilation operators. For the simple harmonic oscillator (without the external force) write

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2\,m\,\omega}}\,\left[\hat{a}\,e^{-i\,\omega\,t} + \hat{a}^\dagger\,e^{i\,\omega\,t}\right]$$

and use commutation relations for $\hat{a}$ and $\hat{a}^\dagger$ ($[\hat{a}, \hat{a}^\dagger] = 1$, all other commutators are zero) to obtain Eq. (1).