Homework Set No. 2, Physics 880.08

Deadline – Wednesday, April 27, 2011

1. (15 pts) Imagine that the full non-perturbative beta-function of QED were

$$\beta(\alpha) \, = \, \frac{\alpha^2}{3\,\pi} \left[1 - e^{1-\frac{1}{\alpha}} \right].$$

Find the running QED coupling constant $\alpha(Q^2)$ for such beta-function. Sketch $\alpha(Q^2)$ as a function of Q^2. Find the UV fixed point and determine the large-Q^2 asymptotics of $\alpha(Q^2)$, i.e., find how it approaches the fixed point.

2. a. (15 pts) Consider a harmonic oscillator in a background of a time-dependent external force (source) $j(t)$. The Lagrangian is

$$L \, = \, \frac{1}{2}\, m\, \dot{q}^2 - \frac{1}{2}\, m\, \omega^2\, q^2 + q\, j(t).$$

Using quasi-classical method for evaluation of path integrals find the time-evolution (Feynman) kernel

$$U(q_f, t_f; q_i, t_i) \, = \, {}_S\langle q_f(t_f)|\, e^{-\frac{i}{\hbar}\,\hat{H}\,(t_f - t_i)}\, |q_i(t_i)\rangle_S \, = \, {}_H\langle q_f, t_f | q_i, t_i\rangle_H$$
$$= \, \int [\mathcal{D}q] \, \exp\left\{ \frac{i}{\hbar} \int\limits_{t_i}^{t_f} dt\, L(t) \right\}.$$

You may use the result derived in class for the harmonic oscillator without the external force, though this time you also need to find the classical action in terms of $j(t)$. In evaluating the classical action assume that $q_i = q_f = 0$, or, more specifically, require that $q_{cl}(t) = 0$ when $j(t) = 0$. When solving classical EOM you may find Fourier-integral decomposition

$$q(t) \, = \, \int\limits_{-\infty}^{\infty} \frac{dE}{2\,\pi}\, e^{-\frac{i}{\hbar}\,E\,t}\, q_E$$

useful.

b. (10 pts) Use the result of part **a** to show that the two-point function for the harmonic oscillator without the external force is given by

$$\langle 0|\, \mathrm{T}\, \hat{q}(t_1)\, \hat{q}(t_2)\, |0\rangle \, = \, i\, \frac{\hbar^2}{m} \int\limits_{-\infty}^{\infty} \frac{dE}{2\,\pi}\, \frac{e^{-\frac{i}{\hbar}\,E\,(t_1 - t_2)}}{E^2 - \hbar^2\,\omega^2 + i\,\epsilon}. \qquad (1)$$

c. (10 pts) Re-derive the two-point function in Eq. (1) by using creation and annihilation operators. For the simple harmonic oscillator (without the external force) write

$$\hat{q}(t) \; = \; \sqrt{\frac{\hbar}{2\, m\, \omega}} \; \left[\hat{a}\, e^{-i\,\omega\, t} + \hat{a}^\dagger\, e^{i\,\omega\, t}\right]$$

and use commutation relations for $\hat{a}$ and $\hat{a}^\dagger$ ($[\hat{a}, \hat{a}^\dagger] = 1$, all other commutators are zero) to obtain Eq. (1).