## Homework Set No. 1, Physics 880.08 Deadline – Monday, October 17, 2011

1. Consider a real scalar interacting field theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \, \partial^{\mu} \phi - \frac{m^2}{2} \, \phi^2 - \frac{\lambda}{3!} \, \phi^3$$

where  $\lambda$  is a real number.

(a) (5 pts) Construct Euler-Lagrange equation for this theory.

(b) (5 pts) Find the energy-momentum tensor  $T^{\mu\nu}$  for this theory and show explicitly that it is conserved,  $\partial_{\mu} T^{\mu\nu} = 0$ , for the fields satisfying Euler-Lagrange equation found in part (a).

2. The Lagrangian density for a two-component complex scalar field

$$\vec{\phi} = \left(\begin{array}{c} \phi_1\\ \phi_2 \end{array}\right)$$

is given by

$$\mathcal{L} = \partial_{\mu}\vec{\phi}^{\dagger} \cdot \partial^{\mu}\vec{\phi} - m^{2}\vec{\phi}^{\dagger}\vec{\phi}$$

where Hermitean conjugation is defined by

$$ec{\phi}^{\,\,\dagger}\,=\,(\phi_1^*,\phi_2^*)$$
 .

(a) (3 pts) Show that the above Lagrangian is invariant under the following global SU(2) symmetry

$$\phi_i \to \phi'_i = \left( \exp\left\{ i \frac{\vec{lpha} \cdot \vec{\sigma}}{2} \right\} \right)_{ij} \phi_j$$

with  $\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$  an arbitrary coordinate-independent vector,  $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$  the Pauli matrices, and i, j = 1, 2. Summation over repeated indices is assumed.

(b) (7 pts) Find the conserved current  $j^a_{\mu}$  and charge  $Q^a$  corresponding to this symmetry (here a = 1, 2, 3).

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3. Consider generators of some Lie group obeying Lie algebra commutation relations

$$[X_a, X_b] = i f_{abc} X_c \tag{1}$$

with anti-symmetric structure constants  $f_{abc}$ .

(a) (5 pts) Prove the Jacobi identity

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0$$

by expanding out the commutators.

(b) (5 pts) Use the commutation relation (1) for  $X_a$ 's in the Jacobi identity to show that

$$f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0,$$

which is also often referred to as the Jacobi identity.

4. (10 pts) Using Gell-Mann matrices (and their commutators) find the structure constants  $f^{147}$  and  $f^{458}$  of the group SU(3).

5. (10 pts) In class we defined the generators of Lorentz group by

$$L_{\mu\nu} = i \left( x_{\mu} \partial_{\nu} - x_{\nu} \partial_{\mu} \right).$$

Show that these generators obey the following algebra

$$[L_{\mu\nu}, L_{\rho\sigma}] = i \eta_{\nu\rho} L_{\mu\sigma} - i \eta_{\mu\rho} L_{\nu\sigma} - i \eta_{\nu\sigma} L_{\mu\rho} + i \eta_{\mu\sigma} L_{\nu\rho}.$$