

Homework Set No. 2, Physics 880.08

Deadline – Friday, October 28, 2011

1. (10 pts) In class we showed that Dirac spinors transform as

$$\psi_D(x) \rightarrow \psi'_D(x') = \begin{pmatrix} e^{-\frac{i}{2} \vec{\sigma} \cdot (\vec{\theta} + i\vec{\xi})} & 0 \\ 0 & e^{-\frac{i}{2} \vec{\sigma} \cdot (\vec{\theta} - i\vec{\xi})} \end{pmatrix} \psi_D(x) \quad (1)$$

under Lorentz transformations. Show that this transformation rule is equivalent to

$$\psi_D(x) \rightarrow \psi'_D(x') = e^{-\frac{i}{4} \omega^{\mu\nu} \sigma_{\mu\nu}} \psi_D(x)$$

where

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu].$$

As usual $\xi^i = \omega^{0i}$ and $\theta_i = \frac{1}{2} \epsilon_{ijk} \omega_{jk}$. You may consider boosts and rotations separately for the full credit.

2. (a) (5 pts) Complete the proof started in class that

$$\bar{\psi} \gamma^\mu \psi$$

is a 4-vector by showing that it transforms like one under infinitesimal boosts. ψ is the Dirac spinor which transforms according to Eq. (1).

- (b) (5 pts) Prove that

$$\bar{\psi} \gamma^\mu \gamma^5 \psi$$

is a 4-vector under both boosts and rotations. What happens to it under parity?

3. Consider a real scalar field theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2 \quad (2)$$

where λ and v are some positive real numbers. Imagine that the theory lives in $1 + 1$ space-time dimensions labeled (t, x) .

- (a) (3 pts) Construct the Hamiltonian for the theory. Show that, for time-independent fields $\phi(t, x) = \phi(x)$, the energy minima (the vacua) of the Hamiltonian are given by

$$\phi = \pm v.$$

(b) (7 pts) Find the time-independent solution of the equations of motion for the Lagrangian (2) that interpolates between the two vacua. That is find the solution $\phi(x)$ satisfying the following conditions

$$\begin{aligned}\phi(x = -\infty) &= -v \\ \phi(x = +\infty) &= v.\end{aligned}$$

(You may also require that $\phi(x = 0) = 0$ for simplicity.) Such solution is known as the 'kink' solution and is the simplest example of a soliton.

4. Consider a massive Dirac field ψ with mass m .

a. (2 pts) Starting with Dirac equation

$$[i\gamma^\mu \partial_\mu - m]\psi = 0$$

derive an equation for $\bar{\psi}$.

b. (3 pts) Using the result of part a show that the electromagnetic current

$$j_\mu = \bar{\psi}\gamma_\mu\psi$$

is conserved at the classical level, i.e., show that $\partial_\mu j^\mu = 0$.

c. (5 pts) Defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ use the anti-commutation relations for γ -matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ to show that

$$\{\gamma^\mu, \gamma^5\} = 0.$$

Use this result to show that the divergence of the axial vector current

$$j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$$

is

$$\partial_\mu j^{5\mu} = 2im\bar{\psi}\gamma^5\psi.$$

That is the axial current is conserved for massless particles (in this classical theory).