

Homework Set No. 4, Physics 880.08

Deadline – Monday, November 28, 2011

1. Time-ordered product of real scalar fields is defined by

$$T\phi(x)\phi(y) = \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x),$$

where ϕ 's are operators in Heisenberg picture.

- a. (5 pts) In a free scalar field theory with mass m use Klein-Gordon equation along with the canonical commutation relations to show that

$$[\partial^2 + m^2] T\phi(x)\phi(y) = -i\delta^4(x - y)$$

where the derivative squared (the D'Alembertian) is taken with respect to 4-coordinates x .

- b. (10 pts) Similar to what we did in class for retarded Green function, find an explicit expression for the Feynman propagator in coordinate space in a massless ($m = 0$) theory by performing the following integral

$$D_F(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)} \frac{i}{k^2 + i\epsilon}.$$

Is Feynman propagator causal?

2. Time-ordered product of Dirac spinors is defined by

$$T\psi_\alpha(x)\bar{\psi}_\beta(y) = \theta(x^0 - y^0)\psi_\alpha(x)\bar{\psi}_\beta(y) - \theta(y^0 - x^0)\bar{\psi}_\beta(y)\psi_\alpha(x)$$

where $\alpha, \beta = 1, 2, 3, 4$ are Dirac indices. ψ and $\bar{\psi}$ are operators in Heisenberg picture.

- a. (3 pts) In a free Dirac field theory with mass m use Dirac equation and canonical anti-commutation relations to show that (α' is summed over)

$$[i\gamma^\mu \partial_\mu - m]_{\alpha\alpha'} T\psi_{\alpha'}(x)\bar{\psi}_\beta(y) = i\mathbf{1}_{\alpha\beta}\delta^4(x - y).$$

- b. (10 pts) Perform explicit calculation to show that the Feynman propagator for fermions is

$$S_F(x - y) \equiv \langle 0 | T\psi(x)\bar{\psi}(y) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)} \frac{i(\gamma \cdot k + m)}{k^2 - m^2 + i\epsilon}.$$

(We did much of this in class, but it is still useful to do by yourself.)

3. a. (3 pts) For a free real scalar field theory with mass m show that

$$[\phi(x), \phi(y)] = \int \frac{d^4 k}{(2\pi)^4} e^{-i k \cdot (x-y)} \text{Sign}(k^0) (2\pi) \delta(k^2 - m^2) \quad (1)$$

where

$$\text{Sign}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$$

Again ϕ 's are operators in Heisenberg picture, x and y are some coordinate space 4-vectors.

b. (2 pts) Verify that Eq. (1) gives correct equal-time commutation relations for the fields

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = 0.$$

c. (5 pts) Using Eq. (1) show that

$$\left(\frac{\partial}{\partial x^0} [\phi(x), \phi(y)] \right)_{y^0=x^0} = -i \delta^3(\vec{x} - \vec{y}),$$

i.e., reproduce another one of the equal-time canonical commutation relations. Also show that

$$[\dot{\phi}(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = 0$$

reproducing the remaining canonical commutation relation.

4. (7 pts) Construct scalar electrodynamics following steps similar to what we did in class in deriving QED. First consider a complex scalar field theory with the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi.$$

It is invariant under a global $U(1)$ symmetry: $\phi(x) \rightarrow e^{i\alpha} \phi(x)$ with α a real number. Gauge this Lagrangian by modifying it to have a *local* $U(1)$ symmetry

$$\phi(x) \rightarrow e^{i\alpha(x)} \phi(x) \quad (2)$$

with the help of a new vector field A_μ which transforms as

$$A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \alpha(x). \quad (3)$$

Finally add the Lagrangian density for the free vector field. What is the resulting Lagrangian? Show explicitly that it is invariant under the local $U(1)$ transformations given by Eqs. (2) and (3).

5.* (Optional problem - do it for extra credit.) (15 pts) For a free real scalar field theory with mass m find the following correlator

$$G_4(x_1, x_2, x_3, x_4) = \langle 0 | T \{ \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \} | 0 \rangle$$

where x_1, x_2, x_3 , and x_4 are four different 4-vectors. Express the answer in terms of the Feynman propagators

$$\begin{aligned} D_F(x - y) &= \int \frac{d^4 k}{(2\pi)^4} e^{-i k \cdot (x - y)} \frac{i}{k^2 - m^2 + i \epsilon} \\ &= \theta(x^0 - y^0) D(x - y) + \theta(y^0 - x^0) D(y - x) \end{aligned}$$

where

$$D(x - y) = \int \frac{d^3 k}{(2\pi)^3 2 E_k} e^{-i k \cdot (x - y)}.$$

(Hint: I found it easier to first consider the case when the time-ordering is, say, $x_1^0 > x_2^0 > x_3^0 > x_4^0$, obtain the answer in this case, and then generalize to other time-orderings by considering permutations of arguments.)