Homework Set No. 6, Physics 8808.02

Deadline – Tuesday, April 9, 2013

1. (10 pts) In class we showed that the two-point Green function in the ϕ^4-theory can be written in terms of functional derivatives as

$$\langle\psi_0|T\phi(x_1)\phi(x_2)|\psi_0\rangle = -\frac{\left\{ \frac{\delta^2}{\delta j(x_1)\,\delta j(x_2)}\, e^{-i\frac{\lambda}{4!}\int d^4x \frac{\delta^4}{\delta j(x)^4}}\, e^{-\frac{1}{2}\int d^4y\, d^4z\, j(y)\, D_F(y-z)\, j(z)} \right\}\Big|_{j=0}}{\left\{ e^{-i\frac{\lambda}{4!}\int d^4x \frac{\delta^4}{\delta j(x)^4}}\, e^{-\frac{1}{2}\int d^4y\, d^4z\, j(y)\, D_F(y-z)\, j(z)} \right\}\Big|_{j=0}}$$

with $D_F(y-z)$ the Feynman propagator for the real scalar field. Evaluate the above expression to the order-λ^2 (in class we did this to order-λ) to find the two-point function to order-λ^2. (Expand the numerator and the denominator separately first, and then divide the obtained expressions.) Compare your answer with that derived in the Autumn semester in class using contractions (page 117 of class notes).

2. a. (5 pts) By explicitly expanding the exponentials on the left-hand-side and carrying out the Grassmann integrals show that the following relation holds

$$\int d\bar{\chi}_1\, d\chi_1\, d\bar{\chi}_2\, d\chi_2\, \exp\left[-\sum_{i,j} a_{ij}\,\bar{\chi}_i\,\chi_j\right] \exp\left[\sum_k (\bar{\chi}_k\,\xi_k + \bar{\xi}_k\,\chi_k)\right] = (\det A)\, \exp\left[\sum_{i,j} \bar{\xi}_i A^{-1}_{ij}\,\xi_j\right]$$

where χ_i and ξ_j are Grassmann variables, and A is a 2×2 Hermitean matrix with elements a_{ij}.

b (5 pts) Show that

$$-i\frac{\partial}{\partial\bar{\xi}}F = \chi\, F = F\,\chi$$
$$i\frac{\partial}{\partial\xi}F = \bar{\chi}\, F = F\,\bar{\chi}$$

for the function

$$F = \exp\left[i\,(\bar{\xi}\,\chi + \bar{\chi}\,\xi)\right].$$

Here χ and ξ are Grassmann variables.

OVER

3. Nature of the perturbation series.

Consider a zero-dimensional "field theory" defined by the "path integral"

$$I(m,\lambda) \;=\; \int\limits_{-\infty}^{\infty} dx\, e^{-S[x]} \tag{1}$$

where the (Euclidean) action is

$$S[x] \;=\; m\, x^2 + \lambda\, x^4.$$

a. (10 pts) Expand $I(m,\lambda)$ in a perturbation series in the powers of λ. Show that the radius of convergence of the series is zero.

You may need the integral definition of the gamma-function

$$\Gamma(z) \;=\; \int\limits_{0}^{\infty} dt\, t^{z-1}\, e^{-t}$$

along with the following property

$$\Gamma(z+1) \;=\; z\,\Gamma(z).$$

b. (10 pts) Let $I_n(m,\lambda)$ denote the truncated perturbation series from part **a** (partial sum) with the highest power of λ being λ^n. Using your favorite numerical software plot $I_n(m=1,\lambda)$ for $n=0,1,2,3,4,5,\ldots$ as functions of λ in the range $\lambda \in [0, 0.1]$ (I got better-looking plots in this range, but you may change the range to make a better picture). On the same plot draw the curve corresponding to the exact result

$$I(m,\lambda) \;=\; \sqrt{\frac{m}{4\,\lambda}}\; e^{\frac{m^2}{8\,\lambda}}\, K_{1/4}\left(\frac{m^2}{8\,\lambda}\right),$$

with $K_{1/4}$ the modified Bessel function. Demonstrate the asymptotic nature of the series: as you increase the order n, the quality of the perturbative approximation first increases, but then rapidly starts to decrease.

c. OPTIONAL (5 pts) Quasi-classical approximation: evaluate the integral $I(m,\lambda)$ in Eq. (1) using the steepest descent (aka saddle point) method. Find the "classical solution" ($x_{cl} = 0$), expand the power of the exponent to quadratic order in fluctuations ξ (where $x = x_{cl} + \xi$), and integrate over all ξ. How good is the approximation? Note that at small-λ the saddle-point approximation works. (This is usually true for field theories too.)

OVER

4. (10 pts) Consider a non-Abelian gauge theory with the gauge field A_μ^a and the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\,\mu\nu}.$$

Here

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

with f^{abc} the structure constants of the gauge group $SU(N)$.

Write the equations of motion for this theory. If we define $J^{a\,\mu}$ by

$$\partial_\nu F^{a\,\nu\mu} = J^{a\,\mu}$$

what is $J^{a\,\mu}$ for the above Lagrangian?