Instantons

Consider $SU(2)$ Yang-Mills theory in Euclidean space: $x_\mu x^\mu = (x^0)^2 + \vec{x}^{\,2}$, $\mu = 1,2,3,4$ ← time

$$A_\mu = \sum_{a=1}^{3} \frac{\tau^a}{2} A_\mu^a \,, \quad F_{\mu\nu} = \sum_{a=1}^{3} \frac{\tau^a}{2} F_{\mu\nu}^a \,, \quad \tau^a \sim \text{Pauli matrices}$$

$$\mathcal{L} = -\frac{1}{2}\, \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$$

redefine $A_\mu^{new} = \frac{g}{i} A_\mu^{old} \Rightarrow \mathcal{L} = \frac{1}{2g^2} \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

Gauge transformation now is

$$A_\mu \to U A_\mu U^{-1} + U \partial_\mu U^{-1}$$

$\Rightarrow$ look for solution of Y-M equations in vacuum $\mathcal{D}_\mu F^{\mu\nu} = 0$

which is localized in space-time:

$$F_{\mu\nu}(x) \xrightarrow[|x| \to \infty]{} 0$$

However, $F_{\mu\nu} = 0$ does not imply $A_\mu = 0$.

The field could be "pure gauge":

$$A_\mu \Big|_{|X|\to\infty} = U \partial_\mu U^{-1}$$

In $A^0 = 0$ gauge one can classify different U's by the winding (Pontryagin) number

$$n_w = \frac{1}{24\pi^2} \int d^3x \; \varepsilon^{ijk} \, \mathrm{Tr}\left[(U^+ \partial_i U)(U^+ \partial_j U)(U^+ \partial_k U) \right]$$

Written in terms of gauge fields it is called Chern-Simons number

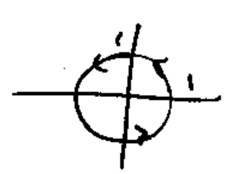
$$n_{cs} = \frac{1}{16\pi^2} \int d^3x \; \varepsilon^{ijk} \left(A_i^a \partial_j A_k^a + \frac{1}{3} \varepsilon^{abc} A_i^a A_j^b A_k^c \right)$$

Winding # counts how many times the manifold is covered.

Example | $f(\theta) = e^{im\theta}$, $0 \leq \theta \leq 2\pi$

$m \sim$ integer

$\Rightarrow n_w = \displaystyle\int_0^{2\pi} \frac{d\theta}{2\pi i} \frac{f'(\theta)}{f(\theta)} \Rightarrow n_w = m$ ~ counts how many times $f(\theta)$ went around the unit circle.



$f(\theta)$ maps a circle (1-sphere) onto $U(1)$ group space.

Example | map 3-sphere onto $SU(2)$ group space: an element of $su(2)$ is

$$U = e^{i\vec{\alpha}\cdot\vec{\tau}}, \quad \vec{\tau} \sim \text{Pauli matrices}$$

Equivalently $$U = \cos|\vec{\alpha}| + i\frac{\vec{\alpha}}{|\vec{\alpha}|}\cdot\vec{\tau}\sin|\vec{\alpha}|$$

$$= u_0 + i\,\vec{u}\cdot\vec{\tau}$$

with $u_0^2 + \vec{u}^2 = 1$ => a 3-sphere.

Define a non-gauge invariant current

$$\boxed{K_\mu = 4\,\varepsilon_{\mu\nu\rho\sigma}\ \mathrm{tr}\left[A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma\right]}$$

One can show that $\partial_\mu K^\mu = 2\,\mathrm{tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$

where the <u>dual field strength</u> is defined

by $\boxed{\tilde{F}_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}}$.

$$2\,\mathrm{tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) = \mathrm{tr}\left(\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\right) =$$

$$= \mathrm{tr}\Big[\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right)\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho + [A_\rho, A_\sigma]\right)\Big] = \mathrm{tr}\Big[\varepsilon^{\mu\nu\rho\sigma}\Big(4(\partial_\mu A_\nu)(\partial_\rho A_\sigma) +$$

$$+ 2\,\partial_\mu A_\nu\,[A_\rho, A_\sigma] + 2\,[A_\mu, A_\nu]\,\partial_\rho A_\sigma + [A_\mu, A_\nu]\cdot[A_\rho, A_\sigma]\Big)\Big] = \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\Big[4\,\partial_\mu\left(A_\nu \partial_\rho A_\sigma\right) +$$

$$+ 4\,\partial_\mu A_\nu \underbrace{[A_\rho, A_\sigma]}_{2 A_\rho A_\sigma} + [A_\mu, A_\nu][A_\rho, A_\sigma]\Big]$$

Now $\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left[[A_\mu, A_\nu][A_\rho, A_\sigma]\right] = \varepsilon^{\mu\nu\rho\sigma}\cdot$

$$A_\mu^a A_\nu^b A_\rho^c A_\sigma^d \quad i f^{abe} \, i f^{cdf} \, tr(T^e T^f)$$

$$= -\varepsilon^{\mu\nu\rho\sigma} \cdot \frac{1}{2} A_\mu^a A_\nu^b A_\rho^c A_\sigma^d \, f^{abe} f^{cde}$$

$$= -\varepsilon^{\mu\nu\rho\sigma} \frac{1}{2} A_\mu^a A_\nu^b A_\rho^c A_\sigma^d \left(\frac{2}{3} (\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}) \right.$$

$$\left. + \left(d^{ace} d^{bde} - d^{bce} d^{ade} \right) \right) = 0$$

as d^{abc} are absolutely symmetric

$$d^{abc} = 2 \, tr\left(T^a \{ T^b, T^c \}\right).$$

We get $2 \, tr\left(F_{\mu\nu} \tilde{F}^{\mu\nu}\right) = 4 \, \varepsilon^{\mu\nu\rho\sigma} \, tr\left[\partial_\mu (A_\nu \partial_\rho A_\sigma)\right.$

$$\left. + 2 \underbrace{(\partial_\mu A_\nu) A_\rho A_\sigma}_{\frac{1}{3}\partial_\mu (A_\nu A_\rho A_\sigma) \text{ cyclic property of the trace}} \right] = \partial_\mu K^\mu \quad \text{, as desired.}$$

$$\Rightarrow \quad n_{CS} = \frac{1}{16\pi^2} \cdot \frac{1}{2} \cdot \int d\sigma_\mu \, K^\mu$$

Define Topological charge (4d Pontryagin index)

$$Q = \frac{1}{16\pi^2} \int d^4x \; tr\left[F_{\mu\nu} \tilde{F}^{\mu\nu}\right]$$

$$\Rightarrow Q = \frac{1}{32\pi^2} \int d^4x \; \partial_\mu K^\mu$$

Euclidean action

$$S_E = \frac{1}{2g^2} \int d^4x \; tr\left(F_{\mu\nu} F^{\mu\nu}\right)$$

$\Rightarrow$ as $\int d^4x \; tr\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right)^2 \geqslant 0 \quad \left(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = F_{\mu\nu}F^{\mu\nu}\right)$

$$\Rightarrow \int d^4x \, tr\left(F_{\mu\nu} F^{\mu\nu} \pm F_{\mu\nu} \tilde{F}^{\mu\nu}\right) \geqslant 0$$

$$\Rightarrow \int d^4x \, tr F_{\mu\nu} F^{\mu\nu} \geqslant \mp \int d^4x \, tr F_{\mu\nu} \tilde{F}^{\mu\nu}$$

$$\Rightarrow S_E = \frac{1}{2g^2} \int d^4x \; tr \; F_{\mu\nu} F^{\mu\nu} \geqslant \underbrace{\frac{1}{2g^2} \left| \int d^4x \, tr F_{\mu\nu} \tilde{F}^{\mu\nu} \right|}_{\frac{1}{2g^2} \cdot 16\pi^2 Q}$$

$$\Rightarrow \boxed{S_E \geqslant \frac{8\pi^2}{g^2} \cdot Q}$$

the minimum of the action is reached when $F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}$

$\Rightarrow$ fields are (anti)-self-dual!

$\Rightarrow$ as the action is minimized $\Rightarrow$ also a solution of YM equations

If we find a solution which falls off fast as $|\vec{x}| \to \infty$ $\Rightarrow$

$$Q = \frac{1}{16\pi^2} \int d^4x \, \mathrm{tr}\left[F_{\mu\nu} \tilde{F}^{\mu\nu}\right] = \frac{1}{32\pi^2} \int d^4x \, \partial_\mu K^\mu$$

$$= \frac{1}{32\pi^2} \int dt \, \frac{\partial}{\partial t} \int d^3x \, K^0 = n_{cs}(t=+\infty) - n_{cs}(t=-\infty)$$

$\Rightarrow$ we may have a field configuration which takes us from a vacuum with one n_{cs} at $t=-\infty$ to another one (with different n_{cs}) at $t=+\infty$ ~ quantum tunneling.

Define 't Hooft symbol $\eta_{a\mu\nu}$ by

$$\eta_{a\mu\nu} = \begin{cases} \varepsilon_{a\mu\nu} & , \ \mu,\nu = 1,2,3 \\ \delta_{a\mu} & , \ \nu = 4 \\ -\delta_{a\nu} & , \ \mu = 4 \end{cases}$$

$a = 1,2,3$ ~ color index. $\eta_{a\mu\nu} = -\eta_{a\nu\mu}$

$$A_\mu^a = 2\,\eta_{a\mu\nu} \frac{x_\nu}{x^2}$$ is a pure gauge solution with $F_{\mu\nu}^a = 0$ everywhere.

(check!)

To construct instanton solution use the ansatz:

$$\boxed{A_\mu^a = 2\eta_{a\mu\nu} \frac{x_\nu}{x^2} f(x^2)}$$

with $f \to 1$ as $x^2 \to \infty$.

Plug this into $F_{\mu\nu}^a = \tilde{F}_{\mu\nu}^a$.

Should get $\boxed{x^2 f'(x^2) - f(1-f) = 0.}$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \overset{su(2)}{\varepsilon^{abc}} A_\mu^b A_\nu^c =$$

$$= 2\eta_{a\nu\rho}\, \partial_\mu \left(\frac{x_\rho}{x^2} f(x^2)\right) - 2\eta_{a\mu\rho}\, \partial_\nu \left(\frac{x_\rho}{x^2} f(x^2)\right) +$$

$$+ \varepsilon^{abc} \cdot 4\, \eta_{b\mu\rho}\, \eta_{c\nu\sigma} \frac{x_\rho}{x^2} \frac{x_\sigma}{x^2} f^2(x^2) = \underline{2\eta_{a\nu\mu}} \frac{f(x^2)}{x^2}$$

$$-2\underline{\eta_{a\mu\nu}} \frac{f}{x^2} - 2\underline{\underline{\eta_{a\nu\rho}}}\, x_\rho \frac{2x_\mu}{(x^2)^2} f + 2\underline{\eta_{a\mu\rho}}\, x_\rho \frac{2x_\nu}{(x^2)^2} f$$

$$+ 4\eta_{a\nu\rho} \frac{x_\rho}{x^2} \cdot x_\mu \cdot f' - 4\eta_{a\mu\rho} \frac{x_\rho}{x^2} x_\nu f' +$$

$$+ 4 \frac{x_\rho x_\sigma}{(x^2)^2} f^2(x^2) \cdot \left[\delta_{\mu\nu} \underbrace{\cancel{\eta_{a\rho\sigma}}}_{0} - \delta_{\mu\sigma} \underline{\underline{\eta_{a\rho\nu}}} + \delta_{\rho\sigma} \underline{\eta_{a\mu\nu}} \right.$$

$$\left. - \delta_{\rho\nu} \underline{\eta_{a\mu\sigma}} \right]$$

where we've used $\varepsilon^{abc} \eta_{b\mu\rho} \eta_{c\nu\sigma} = \delta_{\mu\nu} \eta_{a\rho\sigma}$

$$- \delta_{\mu\sigma} \eta_{a\rho\nu} + \delta_{\rho\sigma} \eta_{a\mu\nu} - \delta_{\rho\nu} \eta_{a\mu\sigma}$$

$$F^a_{\mu\nu} = -4\,\eta_{a\mu\nu}\,\frac{f}{x^2}(1-f) - 4\,\eta_{a\nu\rho}\,\frac{x_\rho x_\mu}{(x^2)^2}\,f(1-f)$$
$$+ 4\,\eta_{a\mu\rho}\,\frac{x_\rho x_\nu}{(x^2)^2}\,f(1-f) + 4\,\eta_{a\nu\rho}\,\frac{x_\mu x_\rho}{x^2}\,f' - 4\,\eta_{a\mu\rho}\,\cdot$$
$$\cdot\,\frac{x_\rho x_\nu}{x^2}\,f'$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,F_{\rho\sigma} = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\,F_{\alpha\beta} =$$

(using $\varepsilon_{\mu\nu\alpha\beta}\,\eta_{a\rho\beta} = \delta_{\rho\mu}\,\eta_{a\nu\alpha} + \delta_{\rho\alpha}\,\eta_{a\mu\nu} + \delta_{\rho\nu}\,\eta_{a\alpha\mu}$)

$$= -2\,\varepsilon_{\mu\nu\alpha\beta}\,\eta_{a\alpha\beta}\,\frac{1}{x^2}\,f(1-f) - 2\,\varepsilon_{\mu\nu\alpha\beta}\,\eta_{a\beta\rho}\,\cdot$$
$$\cdot\,\frac{x_\rho x_\alpha}{(x^2)^2}\,f(1-f) + 2\,\varepsilon_{\mu\nu\alpha\beta}\,\eta_{a\alpha\rho}\,\frac{x_\rho x_\beta}{(x^2)^2}\,f(1-f) +$$
$$+ 2\,\varepsilon_{\mu\nu\alpha\beta}\,\eta_{a\beta\rho}\,\frac{x_\alpha x_\rho}{x^2}\,f' - 2\,\varepsilon_{\mu\nu\alpha\beta}\,\eta_{a\alpha\rho}\,\frac{x_\rho x_\beta}{x^2}\,f'$$

$$= -2\,\underbrace{\left(\eta_{a\nu\mu} + 4\,\eta_{a\mu\nu} + \eta_{a\nu\mu}\right)}_{2\,\eta_{a\mu\nu}}\,\frac{1}{x^2}\,f(1-f) +$$

$$+ 2\left(\delta_{\rho\mu}\,\eta_{a\nu\alpha} + \delta_{\rho\alpha}\,\eta_{a\mu\nu} + \delta_{\rho\nu}\,\eta_{a\alpha\mu}\right)\frac{x_\rho x_\alpha}{(x^2)^2}\,f(1-f)$$

$$- 2\left(\delta_{\rho\mu}\,\eta_{a\nu\beta} + \delta_{\rho\beta}\,\eta_{a\mu\nu} + \delta_{\rho\nu}\,\eta_{a\beta\mu}\right)\frac{x_\rho x_\beta}{(x^2)^2}\,f(1-f)$$

$$- 2\left(\delta_{\rho\mu}\,\eta_{a\nu\alpha} + \delta_{\rho\alpha}\,\eta_{a\mu\nu} + \delta_{\rho\nu}\,\eta_{a\alpha\mu}\right)\frac{x_\alpha x_\rho}{x^2}\,f' \cdot 2$$

$$= \cancel{-4\eta_{a\mu\nu}\frac{1}{x^2}f(1-f)} + 4\eta_{a\nu\alpha}\frac{x_\mu x_\alpha}{(x^2)^2}f(1-f)$$

$$+ \cancel{4\eta_{a\mu\nu}\frac{1}{x^2}f(1-f)} + 4\eta_{a\alpha\mu}\frac{x_\nu x_\alpha}{(x^2)^2}f(1-f)$$

$$-4\eta_{a\nu\alpha}\frac{x_\alpha x_\mu}{x^2}f' - 4\eta_{a\mu\nu}\, f' - 4\eta_{a\alpha\mu}\frac{x_\alpha x_\nu}{x^2}f'$$

$$\Rightarrow \tilde{F}_{\mu\nu} = 4\eta_{a\nu\alpha}\frac{x_\mu x_\alpha}{(x^2)^2}f(1-f) + 4\eta_{a\alpha\mu}\frac{x_\nu x_\alpha}{(x^2)^2}f(1-f)$$

$$-4\eta_{a\nu\alpha}\frac{x_\alpha x_\mu}{x^2}f' - 4\eta_{a\mu\nu}\, f' - 4\eta_{a\alpha\mu}\frac{x_\alpha x_\nu}{x^2}f'$$

Now require $\boxed{F_{\mu\nu} = \tilde{F}_{\mu\nu}}$:

$\eta_{a\mu\nu}$-term: $\quad -\dfrac{f(1-f)}{x^2} = -f' \Rightarrow x^2 f' - f(1-f) = 0$

$\eta_{a\nu\rho}/\eta_{a\nu\alpha}$ term: $\quad -\dfrac{f(1-f)}{(x^2)^2} + \dfrac{f'}{x^2} = \dfrac{f(1-f)}{(x^2)^2} - \dfrac{f'}{x^2}$

$\Rightarrow$ again get $x^2 f' - f(1-f) = 0$.

$\eta_{a\mu\rho}/\eta_{a\mu\alpha}$ terms: $\quad \dfrac{1}{(x^2)^2}f(1-f) - \dfrac{f'}{x^2} = -\dfrac{1}{(x^2)^2}f(1-f)$

$$+ \frac{1}{x^2}f'$$

$\Rightarrow$ the same.

We see that $\boxed{x^2 f' - f(1-f) = 0}$

$$\xi = x^2 \Rightarrow \qquad \xi f' = f(1-f)$$

$$\Rightarrow \frac{df}{f(1-f)} = \frac{d\xi}{\xi} \Rightarrow \ln \xi + \text{const} = \int \frac{df}{f(1-f)}$$

$$= \int df \left(\frac{1}{f} + \frac{1}{1-f} \right) = \ln f - \ln(f-1) = \ln \frac{f}{f-1}$$

$$\Rightarrow \frac{f}{f-1} = c \cdot \xi \Rightarrow f = c\xi(f-1) \Rightarrow$$

$$\Rightarrow f(1 - c\xi) = -c\xi \Rightarrow f = \frac{-c\xi}{1-c\xi} =$$

$$= \frac{\xi}{\xi - \frac{1}{c}} \Rightarrow f(x^2) = \frac{x^2}{x^2+\rho^2}$$

, where $\rho^2 = -\frac{1}{c}$ is the integration constant $\Rightarrow$

$$\boxed{A^a_\mu = \frac{2\eta_{a\mu\nu} x_\nu}{x^2+\rho^2}}$$

$x_\mu A_\mu = 0$ gauge

solves YM equations in vacuum.

$\Rightarrow$ single-<u>instanton</u> solution

(Belavin, Polyakov, Schwartz, Tyupkin '75)

$\Rightarrow \rho \sim$ "size" of instanton, arbitrary.

$\Rightarrow$ this solution corresponds to $Q = 1$

$\Rightarrow$ provides a tunneling transition between a vacuum with CS number n to a vacuum with CS number $n+1$.

$\Rightarrow$ the action is $S_E = \frac{8\pi^2}{g^2}|Q| = \frac{8\pi^2}{g^2}$

$\Rightarrow$ the tunneling amplitude is

$$P_{\text{tunneling}} \propto e^{-S_E} = e^{-\frac{8\pi^2}{g^2}} = e^{-\frac{2\pi}{\alpha_s}}$$

$P \propto e^{-\frac{2\pi}{\alpha_s}}$ ~ not analytic at $\alpha_s = 0$ $\Rightarrow$

$\rightarrow$ non-perturbative!

QCD vacuum may be full of instantons and anti-instantons popping out of the vacuum & dissappearing.

Anti-instanton: $Q = -1$ solution, $\eta_{a\mu\nu} \to \bar{\eta}_{a\mu\nu}$

$$\bar{\eta}_{a\mu\nu} = \begin{cases} \varepsilon_{a\mu\nu}, & \mu,\nu = 1,2,3 \\ -\delta_{a\mu}, & \nu = 4 \\ \delta_{a\nu}, & \mu = 4 \end{cases}$$

~ gives a transition between $n_{cs} \to n_{cs} - 1$ vacua.