Last time: worked with Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}\,[\,i\gamma^\mu \partial_\mu - m\,]\,\psi$$

Symmetry : $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to e^{-i\alpha}\bar{\psi}$ $\Rightarrow$ $j^\mu = \bar{\psi}\gamma^\mu\psi$

is a conserved current $\partial_\mu j^\mu = 0$.

Space-time translation symmetry: $T_{\mu\nu} = i\bar{\psi}\,\gamma_\mu \partial_\nu \psi$

energy-momentum tensor

$$H = \int d^3x \; T_{00} = \int d^3x \; i\psi^+ \partial_t \psi$$ not ≥ 0 !

Def. Helicity ~ projection of spin on direction of motion: $h = \dfrac{\vec{p}\cdot\vec{S}}{|\vec{p}|}$.

χ_R : spin $\uparrow\uparrow$ $\vec{p}$ $\qquad\qquad$ χ_L : spin $\downarrow\uparrow$ $\vec{p}$.

$h = +1/2$ $\qquad\qquad$ $h = -1/2$.

Poincare Group

~ add space-time translations to Lorentz group.

$$x^\mu \to \Lambda^\mu{}_\nu \, x^\nu + a^\mu = x'^\mu$$

$P_\mu = i\partial_\mu$ ~ generators of translations

Poincare algebra

$$\boxed{\begin{aligned} &[P_\mu, P_\nu] = 0 \\ &[P_\mu, J_{\rho\sigma}] = i(g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho) \\ &[J_{\mu\nu}, J_{\rho\sigma}] = \text{same as for Lorentz group.} \end{aligned}}$$

~ classified particles by Poincare group representations.

<u>Very Brief Review:</u>

We know classical field theory by now.

Know that there is a scalar field (spin-∅):

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 \qquad (\varphi \sim \text{real})$$

and Dirac field (spin-½):

$$\mathcal{L} = \bar{\psi} \left[i \gamma^\mu \partial_\mu - m \right] \psi$$

Also heard of a vector field $A_\mu(x)$, though have not seen its Lagrangian. (Same for higher spinors.)

Canonical Quantization

Here we will quantize free fields:

scalar, spinor & vector fields.

We'll start with a real scalar field $\varphi(x)$.

Real Scalar Field.

Earlier in the class you must have seen that if we treat K-G equation as the equation for a single-particle wave function $\varphi(x)$ (just like

Schroedinger eqn, except now relativistic),

then we have a horde of problems:

(i) as the energy $= \pm\sqrt{\vec{k}^2+m^2}$ $\Rightarrow$ can have free particles with negative energy!?

(ii) particle propagation $\langle \vec{x} | e^{-it\hat{H}} | \vec{y} \rangle$ from point $\vec{y}$ to pt. $\vec{x}$ is <u>acausal</u> ~ can find the particle outside the light-cone $\Rightarrow$ it would propagate "faster" than light...

~ in general we know that relativistic kinematics allows for a particle to decay into several particles $\Rightarrow$ particle # is not conserved $\Rightarrow$ should not have a single-particle wave function interpretation.

$\Rightarrow$ we quantize the system treating $\varphi(x)$ as a <u>field</u>!

Again, let us draw an analogy with Quantum Mechanics. Start with a system with degrees of freedom q_i described by Lagrangian $L(q_i, \dot{q}_i)$ $i=1,\dots,N$. Define <u>canonical momenta</u> $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$\Rightarrow$ get Hamiltonian $H = \sum_i \dot{q}_i p_i - L \Rightarrow H(q_i, p_i)$

$\Rightarrow$ quantize by "promoting" q_i, p_i to operators $\hat{q}_i, \hat{p}_i$ such that ($[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ ~ commutator)

$$\boxed{[\hat{q}_i, \hat{p}_j] = i\,\delta_{ij}\,, \quad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0}$$

Mechanics		Field Theory
q_i	$\rightarrow$	$\varphi(x)$
i	$\rightarrow$	x^μ
$p_i = \frac{\partial L}{\partial \dot{q}_i}$	$\rightarrow$	$\pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)}$
$[\hat{q}_i, \hat{p}_j] = i\,\delta_{ij}$	$\rightarrow$	$[\varphi(\vec{x},t), \pi(\vec{x}',t)] = i\,\delta(\vec{x}-\vec{x}')$
$[\hat{q}_i, \hat{q}_j] = 0$	$\rightarrow$	$[\varphi(\vec{x},t), \varphi(\vec{x}',t)] = 0$
$[\hat{p}_i, \hat{p}_j] = 0$	$\rightarrow$	$[\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$.

($\Rightarrow$ Note that canonical quantization favors time direction and is therefore not relat. inv. (physics is indeed invariant).)

$\hat{H}(\hat{q}_i, \hat{p}_i)$	$\rightarrow$	$\hat{H} = \int d^3x \cdot \mathcal{H}(\varphi, \pi)$

Hamiltonian gives time-evolution of the system

We have:

$$[\varphi(\vec{x},t), \pi(\vec{x}',t)] = i\,\delta^3(\vec{x}-\vec{x}')$$

(Equal-time commutation relations.)

$$[\varphi(\vec{x},t), \varphi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$$

where φ, π are operators, $\boxed{\pi(x) \equiv \frac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)}}$

Write

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k}\left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^{+}_{\vec{k}}\, e^{ik\cdot x}\right]$$

(now $\hat{a}$, $\hat{a}^{+}$ are operators)

$$\Rightarrow [\hat{a}_{\vec{k}}, \hat{a}^{+}_{\vec{k}'}] = (2\pi)^3\, 2\mathcal{E}_k\, \delta^3(\vec{k}-\vec{k}')$$

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = [\hat{a}^{+}_{\vec{k}}, \hat{a}^{+}_{\vec{k}'}] = 0$$

$\Leftarrow \begin{cases} \text{for } \mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \dots \\ \pi = \partial_0\varphi = \dot{\varphi} \end{cases}$

(can show)

→ see below

The Hamiltonian is $\boxed{H = \int d^3x \left[\dot{\varphi}(x)\,\pi(x) - \mathcal{L}\right]}$

$$\Rightarrow \quad i\frac{\partial}{\partial t}\hat{O} = [\hat{O}, \hat{H}]$$

Heisenberg picture.

$$\Rightarrow \hat{O}(\vec{x},t) = e^{i\hat{H}t}\,\hat{O}(\vec{x},0)\,e^{-i\hat{H}t}$$

Free scalar field: $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \Rightarrow$

$$\Rightarrow \pi(x) = \frac{\delta\mathcal{L}}{\delta\dot{\varphi}} = \frac{\delta\mathcal{L}}{\delta(\partial_0\varphi)} = \partial_0\varphi \Rightarrow$$

$$H = \int d^3x \left[\dot{\varphi}^2 - \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2\right]$$

define $\overleftrightarrow{\partial_0}$ by $\varphi_1 \overleftrightarrow{\partial_0} \varphi_2 = \varphi_1 \partial_0 \varphi_2 - \varphi_2 \partial_0 \varphi_1$

Note that $\int d^3x \; e^{ik\cdot x} \overleftrightarrow{\partial_0} e^{-ik'\cdot x} = \int d^3x \left[-i\mathcal{E}_{k'} - i\mathcal{E}_k \right]$

$\cdot e^{ix\cdot(k-k')} = -2i\,\mathcal{E}_k (2\pi)^3 \delta(\vec{k}-\vec{k}')$

$\Rightarrow$ if $\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \left[\hat{a}_{\vec{k}} e^{-ik\cdot x} + \hat{a}^+_{\vec{k}} e^{ik\cdot x} \right]$

$$\Rightarrow \int d^3x \; e^{ik'\cdot x} \overleftrightarrow{\partial_0} \varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \hat{a}_{\vec{k}} (-2i\mathcal{E}_k)(2\pi)^3 \delta(\vec{k}-\vec{k}')$$

$$= -i\,\hat{a}_{\vec{k}'}$$

as $\int d^3x \; e^{ik\cdot x} \overleftrightarrow{\partial_0} e^{+ik'\cdot x} = 0$. (why?)

$$\Rightarrow \boxed{\hat{a}_{\vec{k}} = \int d^3x \; e^{ik\cdot x} \; i \overleftrightarrow{\partial_0} \varphi(x)}$$

Similarly $\boxed{\hat{a}^+_{\vec{k}} = \int d^3x \; \varphi(x) \, i \overleftrightarrow{\partial_0} \, e^{-ik\cdot x}}$

$$\Rightarrow \left[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'} \right] = \int d^3x \, d^3y \left[e^{ik\cdot x} i \overleftrightarrow{\partial_0} \varphi(x), \varphi(y) i \overleftrightarrow{\partial_0} e^{-ik'\cdot y} \right]$$

$$= \int d^3x\, d^3y \; e^{ik\cdot x - ik'\cdot y} (-) \left[\pi(x) - i\,\mathcal{E}_k \varphi(x), \; -i\mathcal{E}_{k'} \varphi(y) - \pi(y) \right]$$

$$= \int d^3x\, d^3y \; e^{ik\cdot x - ik'\cdot y} (-i) \left\{ \left[\varphi(\underset{\vec{x},t}{x}), \pi(\underset{\vec{y},t}{y}) \right] \mathcal{E}_k + \left[\varphi(y), \pi(x) \right] \mathcal{E}_{k'} \right\} =$$

$$= \int d^3x\, d^3y \; e^{ik\cdot x - ik'\cdot y} (\mathcal{E}_k + \mathcal{E}_{k'}) \delta(\vec{x}-\vec{y}) = e^{i(\mathcal{E}_k - \mathcal{E}_{k'})\cdot t} (\mathcal{E}_k + \mathcal{E}_{k'}) (2\pi)^3 \delta(\vec{k}-\vec{k}')$$

$= 2\,\mathcal{E}_k\,(2\pi)^3\,\delta(\vec{k}-\vec{k}')$ as advertised!

If $\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k}\left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^{+}_{\vec{k}}\, e^{ik\cdot x}\right]$

then $\pi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k}\left[-i\,\mathcal{E}_k\,\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + i\,\mathcal{E}_k\,\hat{a}^{+}_{\vec{k}}\, e^{ik\cdot x}\right]$.

We'll be working in Heisenberg picture:

$$\left.\begin{aligned}\phi(\vec{x},t) &= e^{iHt}\,\phi(\vec{x},0)\,e^{-iHt}\\ \pi(\vec{x},t) &= e^{iHt}\,\pi(\vec{x},0)\,e^{-iHt}\end{aligned}\right\}\text{– operators are time-dependent}$$

– $|\psi\rangle$ ~ states are time-indep.

There is also Schrödinger picture:

– operators are time-independent: $\phi_S(\vec{x})$, $\pi_S(\vec{x})$

– states are time-dependent $|\psi, t\rangle$.

Interaction picture mixes the two:

$$\phi_I(\vec{x},t) = e^{iH_0 t}\,\phi_S(\vec{x})\,e^{-iH_0 t}$$

$$|a,t\rangle_I = e^{iH_0 t}\,|a,t\rangle_S = e^{iH_0 t}\,e^{-iHt}\,|a\rangle_H$$

H_0 ~ free (non-interacting) Hamiltonian.

in Sch. picture $[\phi(\vec{x}), \pi(\vec{y})] = i\,\delta^3(\vec{x}-\vec{y})$.

in H. picture $i\frac{d\hat{O}}{dt} = [\hat{O}, \hat{H}]$

In general in quantum theory time-evolution is given by Hamiltonian:

$$\boxed{-i\hbar \frac{d}{dt} \langle \psi | \hat{O} | \phi \rangle = \langle \psi | [\hat{H}, \hat{O}] | \phi \rangle.}$$

(comes from generalizing Poisson bracket in QM)

We can separate this equation into two:

$$-i\hbar \frac{d\hat{O}}{dt} = [\hat{H}_1, O] \quad \text{and} \quad i\hbar \frac{d}{dt} |\psi\rangle = \hat{H}_2 |\psi\rangle$$

(+same for $|\phi\rangle$).

where $\hat{H} = \hat{H}_1 + \hat{H}_2$

Heisenberg: $\hat{H}_1 = \hat{H}$, $\hat{H}_2 = 0$

Schrödinger: $\hat{H}_1 = 0$, $\hat{H}_2 = \hat{H}$

Interaction picture: $\hat{H}_1 = \hat{H}_0$, $\hat{H}_2 = \hat{H}_{int.}$

free Ham. interacting Ham.

(for more on these see Sterman, Appendix A.)