Last time:

Canonical commutation relations

$[\varphi(\vec{x},t), \pi(\vec{y},t)] = i\,\delta(\vec{x}-\vec{y}), \ldots$

$\oplus$

Hamiltonian generates time-evolution:

$-i\,\partial_t \varphi = [H, \varphi]$

$-i\,\partial_t \pi = [H, \pi]$

$\Rightarrow$ get Klein-Gordon equation $[\Box + m^2]\varphi = 0$ at the operator level ($\varphi \sim$ operator)

$$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^{+}_{\vec{k}}\, e^{ik\cdot x} \right]$$

Showed that, up to an ∞ constant,

$$H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\, \varepsilon_k\, \hat{N}(\vec{k})$$

where $\hat{N}(\vec{k}) = \hat{a}^{+}_{\vec{k}}\, \hat{a}_{\vec{k}} \sim$ particle # operator.

Net Energy = energy of one particle $\otimes$ # particles

Discussed ground state (vacuum): $\hat{a}_{\vec{k}}|0\rangle = 0$.

One-particle state $|\vec{k}\rangle = \hat{a}^{+}_{\vec{k}}|0\rangle$.

Two-particle state $|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^{+}_{\vec{k}_1}\, \hat{a}^{+}_{\vec{k}_2}|0\rangle$

$\vdots$

N-particle state $|\vec{k}_1, \ldots, \vec{k}_N\rangle = \hat{a}^{+}_{\vec{k}_1} \ldots \hat{a}^{+}_{\vec{k}_N}|0\rangle$

Fock states.

Any state in field theory can be expanded in Fock states

$$|\psi\rangle = C_0|0\rangle + \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} C_{\vec{k}} |\vec{k}\rangle + \int \frac{d^3k_1}{(2\pi)^3 2\varepsilon_{k_1}} \frac{d^3k_2}{(2\pi)^3 2\varepsilon_{k_2}} C_{\vec{k}_1,\vec{k}_2} |\vec{k}_1,\vec{k}_2\rangle + \ldots$$

Take a state $\varphi(x)|0\rangle$. It's one-particle wave function is $\langle\vec{k}|\varphi(x)|0\rangle$.

(As $\langle\vec{k},\vec{k}'\rangle = (2\pi)^3\, 2\varepsilon_k\, \delta(\vec{k}-\vec{k}')$ we call $\langle\vec{k}|\psi\rangle$ a one-particle wave function, $\langle\vec{k}_1,\vec{k}_2|\psi\rangle$ a two-particle wave function, etc.)

$|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^+_{\vec{k}_1} \hat{a}^+_{\vec{k}_2} |0\rangle$ two-particle state

$$H |\vec{k}_1, \vec{k}_2\rangle = (\mathcal{E}_{k_1} + \mathcal{E}_{k_2}) |\vec{k}_1, \vec{k}_2\rangle$$

In general $|\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n\rangle = \hat{a}^+_{\vec{k}_1} \hat{a}^+_{\vec{k}_2} \ldots \hat{a}^+_{\vec{k}_n} |0\rangle$

n-particle state. (Fock states)

Any state of the theory can be expanded into Fock states:

$$\boxed{|\Psi\rangle = c_0 |0\rangle + \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} c_{\vec{k}} |\vec{k}\rangle + \int \frac{d^3k_1\, d^3k_2}{(2\pi)^3 2\mathcal{E}_{k_1} (2\pi)^3 2\mathcal{E}_{k_2}} \cdot c_{\vec{k}_1, \vec{k}_2} |\vec{k}_1, \vec{k}_2\rangle + \ldots}$$

One-particle wave function:

$$\Psi(x) \equiv \langle 0| \varphi(x) |\vec{k}\rangle$$

$$\Psi(x) = \langle 0| \int \frac{d^3p}{(2\pi)^3 2\mathcal{E}_p} \left[\hat{a}_{\vec{p}} e^{-ip\cdot x} + \underbrace{\cancel{\hat{a}^+_{\vec{p}} e^{ip\cdot x}}}_{0}\right] \underbrace{|\vec{k}\rangle}_{\hat{a}^+_{\vec{k}}|0\rangle}$$

$$= \langle 0| \int \frac{d^3p}{(2\pi)^3 2\mathcal{E}_p} e^{-ip\cdot x} \underbrace{[\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{k}}]}_{(2\pi)^3 2\mathcal{E}_p \delta^3(\vec{k}-\vec{p})} |0\rangle = e^{-ik\cdot x}$$

just a plane wave, as expected in free field theory.

Complex Scalar Field

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi$$ ~ the Lagrangian (charged scalar field)

Obeys Klein-Gordon equation classically

$$[\Box + m^2]\varphi = 0 \qquad \left(\frac{\delta \mathcal{L}}{\delta \varphi^*} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi^*)} = 0 \right)$$

General solution:

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k} \left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{b}^+_{\vec{k}}\, e^{ik\cdot x} \right]$$

as φ is now complex $\Rightarrow$ $\hat{a}_{\vec{k}} \neq \hat{b}_{\vec{k}}$.

$$\varphi^+(x) = \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k} \left[\hat{a}^+_{\vec{k}}\, e^{ik\cdot x} + \hat{b}_{\vec{k}}\, e^{-ik\cdot x} \right]$$

Canonical momenta:

$$\pi_\varphi = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}} = \dot{\varphi}^* \quad ; \quad \pi_{\varphi^*} = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}^*} = \dot{\varphi}.$$

Demand that according to the rules of canonical quantization:

$$[\varphi(\vec{x},t),\ \pi_\varphi(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y})$$
$$[\varphi^*(\vec{x},t),\ \pi_{\varphi^*}(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y})$$

all other commutators are zero (equal time)

One can show that for $\hat{a}_{\vec{k}}$ & $\hat{b}_{\vec{k}}$ this means:

$$\left[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}\right] = (2\pi)^3 \, 2\varepsilon_k \, \delta(\vec{k}-\vec{k}')$$

$$\left[\hat{b}_{\vec{k}}, \hat{b}^+_{\vec{k}'}\right] = (2\pi)^3 \, 2\varepsilon_k \, \delta(\vec{k}-\vec{k}')$$

(all other commutators vanish)

The Hamiltonian is

$$H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \varepsilon_k \left[\hat{a}^+_{\vec{k}} \, a_{\vec{k}} + \hat{b}^+_{\vec{k}} \, \hat{b}_{\vec{k}}\right]$$

(after dropping ∞)

$\hat{a}^+_{\vec{k}}$ ~ creates particles

$\hat{b}^+_{\vec{k}}$ ~ -"- anti-particles (opposite charge, same mass)

Vacuum state: $\hat{a}_{\vec{k}}|0\rangle = \hat{b}_{\vec{k}}|0\rangle = 0$.

$\hat{a}^+_{\vec{k}_1} \hat{b}^+_{\vec{k}_2}|0\rangle$ ~ state with 1 particle & 1 anti-particle.

Lagrangian is invariant under

$$\varphi \to e^{i\alpha}\varphi, \quad \varphi^* \to e^{-i\alpha}\varphi$$

α ~ real #.

$$j^\mu = i\left[\varphi\, \partial^\mu \varphi^* - \varphi^*\, \partial^\mu \varphi\right]$$

conserved current.

Conserved charge

$$Q = \int d^3x \, j^0(\vec{x},t) = \int d^3x \;\; i\left[\varphi \underbrace{\partial_t \varphi^*}_{\pi_\varphi} - \varphi^* \underbrace{\partial_t \varphi}_{\pi_{\varphi^*}}\right].$$

$$Q = i\int d^3x \left[\varphi \;\; \pi_\varphi - \varphi^* \; \pi_{\varphi^*}\right]$$

Plug in the fields to get:

$$Q = \int \frac{d^3k}{(2\pi)^3 \, 2\varepsilon_k} \left[\underbrace{\hat{a}^+_{\vec{k}} \; \hat{a}_{\vec{k}}}_{\substack{\#\text{ particles} \\ \text{with charge} \\ +1}} - \underbrace{\hat{b}^+_{\vec{k}} \; \hat{b}_{\vec{k}}}_{\substack{\#\text{ particles} \\ \text{with charge} \\ -1}} \right]$$

<u>Example:</u> π^+ & π^- ~ scalar particles with ± 1 electric charge.

Quantization of Spinor Field

$$\mathcal{L} = \bar{\psi}\,[\,i\gamma^\mu \partial_\mu - m\,]\,\psi$$

First of all, let us change the representation: we will go from Weyl to Dirac representation:

$$\psi_{\text{Weyl}} \to S\,\psi_{\text{Weyl}}, \quad \text{where } S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ in } 2\times 2 \text{ notation}$$

$$\psi_{\substack{\text{Dirac}\\\text{rep}}} = S\,\psi_{\substack{\text{weyl}\\\text{rep}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R + \chi_L \\ \chi_R - \chi_L \end{pmatrix}.$$

$$\gamma^\mu_{\text{Weyl}} \to \gamma^\mu_{\text{Dirac}} = S\,\gamma^\mu_{\text{Weyl}}\,S^{-1} = S\,\gamma^\mu_{\text{weyl}}\,S^T$$

$$\gamma^0_{\text{Weyl}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i_{\text{weyl}} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \Rightarrow$$

$$\gamma^0_{\text{Dirac}} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \gamma^0_{\text{Dirac}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \left[\, \mathcal{L} = \underbrace{\bar{\psi}_W S^T}_{\bar{\psi}_D} S\,[i\gamma^\mu \partial_\mu - m]\,S^T \underbrace{S\psi_W}_{\psi_D} = \bar{\psi}_{Dir}\,[i\gamma^\mu_D \partial_\mu - m]\,\psi_D. \,\right]$$

$$\gamma^i_{\text{Dirac}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \sigma^i & \sigma^i \\ -\sigma^i & \sigma^i \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2\sigma^i \\ -2\sigma^i & 0 \end{pmatrix} = \gamma^i_{Weyl}$$

$$\Rightarrow \boxed{\gamma^0_{Dirac} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; ; \; \gamma^i_{Dirac} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}.$$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in either representation (chirality operator)

$\gamma^5_{Dirac} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. ~ different from Weyl basis.

$$\mathbb{P}_L = \frac{\mathbb{1} - \gamma^5}{2}, \quad \mathbb{P}_R = \frac{\mathbb{1} + \gamma^5}{2}$$

$$\Rightarrow P_L = \frac{1}{2}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \Rightarrow P_L \psi_{Dirac} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \chi_R + \chi_L \\ \chi_R - \chi_L \end{pmatrix} =$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 2\chi_L \\ -2\chi_L \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_L \\ -\chi_L \end{pmatrix}$$ ~ "removes" χ_R from ψ_{Dirac}, leaving only χ_L.

$$P_R = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow P_R \psi_{Dirac} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \chi_R + \chi_L \\ \chi_R - \chi_L \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 2\chi_R \\ 2\chi_R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix}$$ ~ leaves only χ_R.

(all γ-matrix formulas work in both bases.)

Peskin ~ Weyl representation, Ryder ~ Dirac representation

$\gamma^5 \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_L \\ -\chi_L \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} \chi_L \\ -\chi_L \end{pmatrix} \Rightarrow$ chirality -1 ; $\gamma^5 \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix} \Rightarrow$ chirality $+1$.

Solution of Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \Rightarrow \text{apply } i\gamma^\nu\partial_\nu \quad \Rightarrow$$

$$\Big[-\underbrace{\gamma^\nu\partial_\nu\gamma^\mu\partial_\mu}_{\parallel} - m\, i\gamma^\nu\partial_\nu\Big]\psi = 0$$

$$\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu = \frac{1}{2}\{\gamma^\nu,\gamma^\mu\}\partial_\mu\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu$$

$$\Rightarrow \Big[-\partial_\mu\partial^\mu - m\,\underbrace{i\gamma^\mu\partial_\mu}_{\overset{\parallel}{m}}\Big]\psi = 0$$

m by Dirac equation

$$\Rightarrow \boxed{[\partial_\mu\partial^\mu + m^2]\psi = 0}$$

$\Rightarrow$ if the field satisfies Dirac equation, it also satisfies Klein-Gordon equation! ($\Rightarrow$ also has < 0 energy "particles")

$\Rightarrow$ Write the solution as

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k}\Big[e^{-ik\cdot x}\psi^{(+)}(\vec{k}) + e^{ik\cdot x}\psi^{(-)}(\vec{k})\Big]$$

& plug back into the original Dirac equation:

$\partial_\mu \to -ik_\mu$ in the 1st term, $+ik_\mu$ in the second

$$\Rightarrow \text{get} \quad (\gamma\cdot k - m)\psi^{(+)}(\vec{k}) = 0$$

$$(\gamma\cdot k + m)\psi^{(-)}(\vec{k}) = 0$$

$$\Rightarrow \text{write} \quad \psi^{(+)} = \begin{pmatrix}\psi_u^{(+)}\\ \psi_\ell^{(+)}\end{pmatrix}$$

$$\Rightarrow \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \Rightarrow$$

$$\gamma \cdot k = \gamma^0 k_0 + \gamma^i k_i = \gamma_0 k_0 - \vec{\gamma} \cdot \vec{k} = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$$

$$- \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{k} \\ -\vec{\sigma} \cdot \vec{k} & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon & -\vec{\sigma} \cdot \vec{k} \\ \vec{\sigma} \cdot \vec{k} & -\varepsilon \end{pmatrix}$$

$$\Rightarrow (\gamma \cdot k - m) \psi^{(+)} = \begin{pmatrix} \varepsilon - m & -\vec{\sigma} \cdot \vec{k} \\ \vec{\sigma} \cdot \vec{k} & -\varepsilon - m \end{pmatrix} \begin{pmatrix} \psi_u^{(+)} \\ \psi_\ell^{(+)} \end{pmatrix} = 0$$

$$\begin{cases} (\varepsilon - m) \psi_u^{(+)} - \vec{\sigma} \cdot \vec{k} \, \psi_\ell^{(+)} = 0 \\ \vec{\sigma} \cdot \vec{k} \, \psi_u^{(+)} - (\varepsilon + m) \psi_\ell^{(+)} = 0 \end{cases} \Rightarrow \psi_\ell^{(+)} = \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \psi_u^{(+)}$$

solves the whole thing (why?)

$$\Rightarrow \psi^{(+)} = \begin{pmatrix} \psi_u^{(+)} \\ \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \psi_u^{(+)} \end{pmatrix}$$

$\Rightarrow$ reduced a 4-component unknown spinor to 2 unknown components

Similarly $$\psi^{(-)} = \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \psi_\ell^{(-)} \\ \psi_\ell^{(-)} \end{pmatrix}.$$

Choose a basis: $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and

Define
$$\boxed{u_r(\vec{k}) = \sqrt{\varepsilon + m} \begin{pmatrix} \chi_r \\ \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \chi_r \end{pmatrix} ; \quad v_r(\vec{k}) = \sqrt{\varepsilon + m} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \chi_r \\ \chi_r \end{pmatrix}} \quad r = 1, 2$$