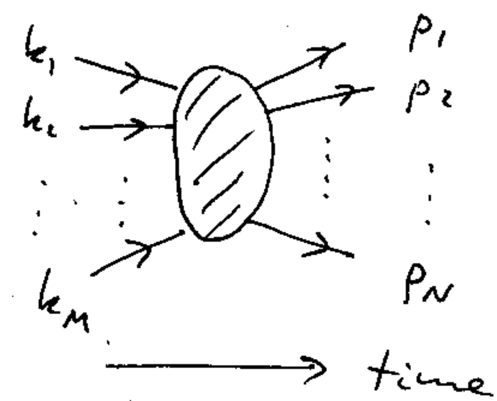
# Cross Sections, $S$-matrix and Reduction Formulas.

## Cross Section & $S$-matrix

Consider a general scattering process:

M particles scatter into N particles.




$|\psi_i\rangle \sim$ initial state at $t = -\infty$.

$|\psi_f\rangle \sim$ a general final state of field th'y at $t = +\infty$.

**Def.** $S$-matrix is the time-evolution operator defined by $\boxed{|\psi_f\rangle = S\,|\psi_i\rangle}$. (aka scattering matrix)

The state norm is conserved: $\langle\psi_f|\psi_f\rangle = \langle\psi_i|\psi_i\rangle$

$\Rightarrow \boxed{S\,S^\dagger = \mathbb{1}} \sim$ Unitary.

In the interaction picture

$$\left.\begin{aligned} |\psi_i\rangle &= |\psi(t=-\infty)\rangle_I \\ |\psi_f\rangle &= |\psi(t=+\infty)\rangle_I \end{aligned}\right\} \Rightarrow \text{as } |\psi(t=+\infty)\rangle_I = U(+\infty, -\infty)\,|\psi(t=-\infty)\rangle_I$$

$$\Rightarrow \boxed{S = U(+\infty, -\infty)}$$

$$|\psi_f\rangle = S\,|\psi_i\rangle = \left[\mathbb{1} + (S - \mathbb{1})\right]|\psi_i\rangle =$$

$$= \underbrace{|\psi_i\rangle}_{\text{old initial state}} + \underbrace{(S-\mathbb{1})\cdot|\psi_i\rangle}_{\text{modification to the state}}$$

$\Rightarrow$ all interactions are contained in $S-1$.

**Def.** T-matrix is defined by $\boxed{S = \mathbb{1} + iT}$.

As $SS^+ = \mathbb{1} \Rightarrow (\mathbb{1}+iT)(\mathbb{1}-iT^+) = \mathbb{1}$

$\Rightarrow i(T - T^+) + TT^+ = 0$

$\Rightarrow \boxed{-i(T - T^+) = TT^+}$ this formula leads to optical theorem.

Due to energy-momentum conservation we can always write (extracting the $\delta$-ftn):

**Def.** $\langle\{p_i\}|\,S - 1\,|\{k_j\}\rangle = \langle\{p_i\}|\,iT\,|\{k_j\}\rangle =$

$$= (2\pi)^4\, \delta^{(4)}\left(\sum_{i=1}^{N} p_i - \sum_{j=1}^{M} k_j\right)\cdot iM\left(\{k_j\}\to\{p_i\}\right).$$

$M$ is the invariant matrix element, a.k.a. the <u>scattering amplitude</u>.

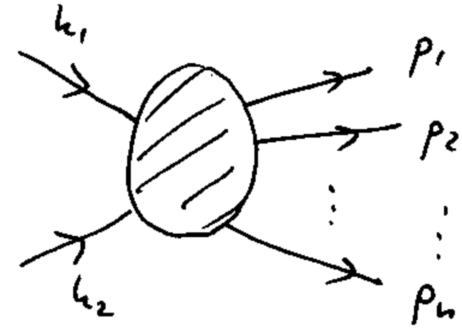
The scattering amplitude contains all dynamical information about scattering.

$|\{k_j\}\rangle \approx$ m-particle state in the interaction picture with $t_0 = -\infty$.

$|\{p_i\}\rangle \approx$ n-particle state —"— with $t_0 = +\infty$.

Consider $2 \to n$ scattering for simplicity:

Scattering amplitude is defined by




$\langle p_1, \dots, p_n | \, i T \, | k_1, k_2 \rangle =$

$$= (2\pi)^4 \, \delta^{(4)}\Big(k_1 + k_2 - \sum_{i=1}^{n} p_i\Big) \cdot \mathcal{M}\,(k_1, k_2 \to \{p_i\}).$$

The initial state now is a 2-particle state:

$$|\psi_i^2\rangle = \int \frac{d^3 q_1 \; d^3 q_2}{(2\pi)^3 2\mathcal{E}_{q_1} (2\pi)^3 2\mathcal{E}_{q_2}} \; f_1(q_1) \, f_2(q_2) \;\; |q_1, q_2\rangle$$

with $f_1(q_1)$ and $f_2(q_2)$ momentum-space wave functions such that $|f_1|^2(q_1)$ & $|f_2|^2(q_2)$ are peaked around $k_1$ & $k_2$ (wave packets).

Now, $|\psi_i\rangle$ is the state at $t = -\infty$. At $t = +\infty$ it becomes $S\,|\psi_i\rangle$, which could be a 2, 3, 4, ...-particle state ~ this is a superposition of Fock states.

The probability to have $n$ particles (with 4-momenta $p_1, p_2, \dots, p_n$) is (dropping $\mathbb{1}$, i.e., requiring some interactions to happen):

$$\boxed{P_{2\to n} = \int \big| \langle p_1, \dots, p_n | \, iT \, | \psi_i^2 \rangle \big|^2 \cdot \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 \, 2\mathcal{E}_i}}$$

(we integrate over $p_1 \dots p_n$ for now).

**Def.** Cross section $(\sigma) = \dfrac{\text{event probability per unit volume and time}}{(\text{target density}) \times (\text{incident flux of particles})}$

To find the cross section let's start with event prob.:

$$P_{2\to n} = \int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 2\varepsilon_i} \left| \langle p_1, \dots, p_n | iT | \psi_i^2 \rangle \right|^2 = \int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 2\varepsilon_i} \cdot$$

$$\cdot \int \frac{d^3q_1\, d^3q_2\, d^3q_1'\, d^3q_2'}{(2\pi)^3 2\varepsilon_{q_1} (2\pi)^3 2\varepsilon_{q_2} (2\pi)^3 2\varepsilon_{q_1'} (2\pi)^3 2\varepsilon_{q_2'}} f_1(q_1) f_2(q_2) f_1^*(q_1') f_2^*(q_2') \cdot$$

$$\cdot (2\pi)^4 \delta^{(4)}\Big(q_1 + q_2 - \sum_{j=1}^{n} p_j\Big) M(q_1, q_2 \to \{p_i\}) \cdot (2\pi)^4 \delta^{(4)}\Big(q_1' + q_2' -$$

$$- \sum_{k=1}^{n} p_k\Big) M^*(q_1', q_2' \to \{p_i\}) =$$

as $f_1$ & $f_2$ are strongly peaked at $k_1$ & $k_2$ $\Rightarrow$ replace $q_1, q_2 \to k_1, k_2$ in $M$'s & delta-ftn $q_1', q_2'$

$$= \int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 2\varepsilon_i} (2\pi)^4 \delta\Big(k_1 + k_2 - \sum_{j=1}^{n} p_j\Big) \left| M(k_1, k_2 \to \{p_i\}) \right|^2 \int \frac{d^3q_1\, d^3q_2}{(2\pi)^{12} \cdot 2\varepsilon_{q_1} 2\varepsilon_{q_2}}$$

$$\cdot \frac{d^3q_1'\, d^3q_2'}{2\varepsilon_{q_1'}\, 2\varepsilon_{q_2'}} \underbrace{\int d^4x\; e^{-i x \cdot (q_1 + q_2 - q_1' - q_2')}}_{\text{used to be } (2\pi)^4 \delta^{(4)}(q_1 + q_2 - q_1' - q_2')} f_1(q_1) f_2(q_2) f_1^*(q_1') f_2^*(q_2')$$

$$= \int d^4x\, |\tilde{f}_1(x)|^2 \, |\tilde{f}_2(x)|^2 \int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 2\varepsilon_i} |M|^2 (2\pi)^4 \delta\Big(k_1 + k_2 - \sum_{j=1}^{n} p_j\Big).$$

$\Rightarrow$ Essentially we traded the $2^{nd}$ $\delta$-function for $d^4x$.

$$\tilde{f}_i(x) = \int \frac{d^3q}{(2\pi)^3 2\varepsilon_q} f_i(q)\, e^{-iq\cdot x} \quad \sim \text{we defined this.}$$

$i = 1, 2$

Event probability per unit volume and time is

$$\boxed{\frac{d P_{2\to n}}{d^4 x} = |\tilde{f}_1(x)|^2 \, |\tilde{f}_2(x)|^2 \int \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2\mathcal{E}_i} \, |M(k_1, k_2 \to \{p_j\})|^2 \cdot (2\pi)^4 \, \delta^{(4)}\Big(k_1 + k_2 - \sum_{j=1}^{n} p_j\Big)}$$

Suppose particle $k_2$ is at rest (target) and $k_1$ is incident on it. The target density is found by noting that the total # of particles in state

$$|\psi_2\rangle = \int \frac{d^3 q}{(2\pi)^3 2\mathcal{E}_q} \, f_2(q) \, |q\rangle$$

(we can factor $|q_1, q_2\rangle = |q_1\rangle |q_2\rangle$ as they are far apart initially).

is

$$N_2 = \langle \psi_2 | \hat{N} | \psi_2 \rangle = \int \frac{d^3 k}{(2\pi)^3 2\mathcal{E}_k} \langle \psi_2 | \hat{a}^+_{\vec{k}} \, \hat{a}_{\vec{k}} | \psi_2 \rangle =$$

$$= \int \frac{d^3 k}{(2\pi)^3 2\mathcal{E}_k} \int \frac{d^3 q}{(2\pi)^3 2\mathcal{E}_q} \frac{d^3 q'}{(2\pi)^3 2\mathcal{E}_{q'}} \, f_2(q) \, f_2^*(q') \, \langle 0 | \underbrace{\hat{a}_{\vec{q}'} \, \hat{a}^+_{\vec{k}}}_{(2\pi)^3 2\mathcal{E}_k \delta(\vec{q}' - \vec{k})} \, \hat{a}_{\vec{k}} \cdot$$

$$\cdot \hat{a}^+_{\vec{q}} |0\rangle = \int \frac{d^3 k}{(2\pi)^3 2\mathcal{E}_k} \frac{d^3 q}{(2\pi)^3 2\mathcal{E}_q} \frac{d^3 q'}{(2\pi)^3 2\mathcal{E}_{q'}} \, f_2(q) \, f_2^*(q') \cdot$$

$$\cdot (2\pi)^3 \, 2\mathcal{E}_{q'} \, \delta(\vec{q}' - \vec{k}) \, (2\pi)^3 \, 2\mathcal{E}_q \, \delta(\vec{q} - \vec{k}) = \int \frac{d^3 q}{(2\pi)^3 2\mathcal{E}_q} \frac{d^3 q'}{(2\pi)^3 2\mathcal{E}_{q'}} \cdot$$

$q' \approx k_2$ as $f_2(q)$ is peaked at $k_2$; $\vec{k} = \vec{q}$

$$\cdot (2\pi)^3 \, 2\mathcal{E}_{k_2} \, \delta(\vec{q} - \vec{q}') \, f_2(q) \, f_2^*(q') = \int d^3 x \cdot 2\mathcal{E}_{k_2} |\tilde{f}_2(x)|^2$$

$$\delta(\vec{q} - \vec{q}') = \int \frac{d^3 x}{(2\pi)^3} e^{i\vec{x}\cdot(\vec{q} - \vec{q}')}, \qquad e^{i\vec{x}\cdot(\vec{q} - \vec{q}')} = e^{-i x \cdot (q - q')}$$