Last time | The Callan-Symanzik Equation

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu}\right] \mathcal{M}\left(Q^2/\mu^2, \alpha_\mu\right) = 0$$ Callan-Symanzik

Solved it with the running coupling

$$\alpha(Q^2) \equiv \rho^{-1}\left(\ln \frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right), \quad \rho(\alpha) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')}$$

$\Rightarrow \mathcal{M}(1, \alpha(Q^2))$ is always a solution of CS eqn.

We worked out the case when $\beta(\alpha) = \beta_2 \alpha^2 \Rightarrow$

got $$\alpha(Q^2) = \frac{\alpha_\mu}{1 - \alpha_\mu \beta_2 \ln Q^2/\mu^2}$$ as before for QED, φ^4, ...

If $\beta_2 > 0$ get ; if $\beta_2 < 0$ get

QED, φ^4

QCD.

RG: general discussion

LSZ : $\langle p_1 \ldots p_{n-2} | S | k_1, k_2 \rangle \sim Z^{-n/2} G^{(n)}$

$$\Rightarrow \mu^2 \frac{d}{d\mu^2}\left[Z^{-n/2} G^{(n)}\right] = 0$$

$\Rightarrow$ obtained full Callan-Symanzik equ:

$$\boxed{\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu} - n\,\gamma(\alpha_\mu) + m_\mu^2\, \gamma_m(\alpha_\mu)\frac{\partial}{\partial m_\mu^2}\right] G^{(4)} = 0}$$

where $\gamma(\alpha_\mu) = \mu^2 \dfrac{d \ln \sqrt{Z}}{d\mu^2}$, $\gamma_m(\alpha_\mu) = \dfrac{1}{m_\mu^2}\mu^2 \dfrac{d m_\mu^2}{d\mu^2}$,

$\beta(\alpha_\mu) = \mu^2 \dfrac{d\alpha_\mu}{d\mu^2}$.

Functional Integral Quantization.

Path Integral Quantum Mechanics

Consider a non-relativistic one-particle quantum mechanics. The particle, at the classical level, has coordinate q and momentum p: the Lagrangian is

$$\boxed{L = \frac{1}{2} m \dot{q}^2 - V(q)}$$

The momentum $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$ => the Hamiltonian is $H = p\dot{q} - L = \frac{p^2}{m} - L = \frac{p^2}{2m} + V(q)$

$$\boxed{H = \frac{p^2}{2m} + V(q)}.$$

Quantize the system by promoting p, q into operators $\hat{p}, \hat{q}$ with the commutation relation

$$\boxed{[\hat{q}, \hat{p}] = i\hbar}.$$

(we put $\hbar$ back in for this topic)

Work in Schrödinger picture: $-i\hbar \frac{d\hat{O}_s}{dt} = 0$ =>

=> operators are time-independent, $i\hbar \frac{d}{dt} |\psi(t)\rangle_s = \hat{H} |\psi(t)\rangle_s$

$\Rightarrow$ $|\psi(t)\rangle_s = e^{-i\frac{\hat{H}}{\hbar}(t-t_0)} |\psi(t_0)\rangle_s$ is the time-dependence of states. ($\hat{H}$ is time-independent in Schrödingerpicture.)

The time-evolution operator is

$$|\psi(t)\rangle_s = U(t, t_0)\, |\psi(t_0)\rangle_s$$

$$\Rightarrow \boxed{U(t, t_0) = e^{-i\frac{\hat{H}}{\hbar}(t-t_0)}} \text{ in Schrödinger picture.}$$

Using single-particle position space states

$$|q(t)\rangle_s \text{ by } \hat{q}\,|q(t)\rangle_s = q(t)|q(t)\rangle_s$$

define the <u>wave function</u> by

$$\boxed{\psi(q,t) = {}_s\langle q(t)|\psi(t)\rangle_s}$$

$$\psi(q,t) = {}_s\langle q(t)|\psi(t)\rangle_s = \langle q(t)|\, e^{-i\frac{\hat{H}}{\hbar}(t-t')} |\psi(t')\rangle_s$$

$$= \int_{-\infty}^{\infty} dq' \cdot \langle q(t)|\, e^{-i\frac{\hat{H}}{\hbar}(t-t')} |q'(t')\rangle \underbrace{\langle q'(t')|\psi(t')\rangle_s}_{\psi(q',t')}$$

$$\Rightarrow \boxed{\psi(q,t) = \int_{-\infty}^{\infty} dq' \langle q(t)|\, e^{-i\frac{\hat{H}}{\hbar}(t-t')} |q'(t')\rangle\, \psi(q',t')}$$

Def. Feynman kernel

$$\boxed{U(q,t;q',t') \equiv {}_S\langle q(t)|\, e^{-i\frac{\hat{H}}{\hbar}(t-t')}\, |q'(t')\rangle_S}$$

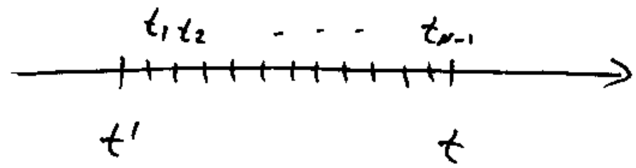
matrix element of the evolution operator.

$\Rightarrow$ $U(q,t;q',t')$ determines time evolution of wave functions.

Let us find $U(q,t;q',t')$: to do this split the time interval in small segments:

N segments

$$\delta t = \frac{t-t'}{N}$$



$$\Rightarrow U(q,t;q',t') = \lim_{N\to\infty} {}_S\langle q(t)|\left[1 - i\frac{\hat{H}}{\hbar}\delta t\right]^N |q'(t')\rangle_S$$

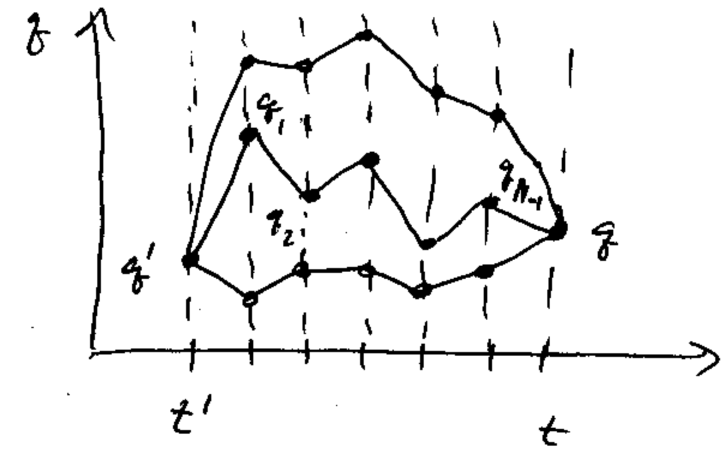
Insert $\mathbb{1} = \int_{-\infty}^{\infty} dq_i(t_i)\cdot |q_i(t_i)\rangle_S\, {}_S\langle q_i(t_i)|$ between each pair of brackets $\Rightarrow$ get

$$U(q,t;q',t') = \lim_{N\to\infty}\int_{-\infty}^{\infty} dq_1\, dq_2 \dots dq_{N-1}\; {}_S\langle q(t)|\, 1 - \frac{i}{\hbar}\hat{H}\delta t\,|q_{N-1}(t_{N-1})\rangle_S$$

$${}_S\langle q_{N-1}(t_{N-1})|\, 1 - \frac{i}{\hbar}\hat{H}\delta t\,|q_{N-2}(t_{N-2})\rangle_S \;\dots\; {}_S\langle q_1(t_1)|\, 1 - \frac{i}{\hbar}\hat{H}\delta t\,|q'(t')\rangle_S .$$

Now we need to evaluate each matrix element in the product in the integrand.

Physical meaning of the integral over all q_i's:
the integral over all possible trajectories (not classical though):



$$ {}_S\langle q_i(t_i)|\, 1 - \frac{i}{\hbar}\hat H(\hat p,\hat q)\,\delta t\,|\, q_{i-1}(t_{i-1})\rangle_S = \int_{-\infty}^{\infty} dp_i \cdot $$

$$ \cdot\, {}_S\langle q_i(t_i)|\, 1 - \frac{i}{\hbar}\hat H(\hat p,\hat q)\,\delta t\,|\, p_i(t_i)\rangle_S \; {}_S\langle p_i(t_i)| q_{i-1}(t_{i-1})\rangle_S $$

$$ = \text{using } \hat H = \frac{\hat p^2}{2m} + V(\hat q) \quad = \int_{-\infty}^{\infty} dp_i \cdot \left[1 - \frac{i}{\hbar} H(p_i, q_i)\,\delta t\right] \cdot $$

$$ \cdot\, \frac{1}{2\pi\hbar}\, e^{\frac{i}{\hbar} p_i (q_i - q_{i-1})} $$

as $\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar} pq}$.

$$ \Rightarrow U(q,t;q',t') = \lim_{N\to\infty} \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N-1} \frac{dq_i\, dp_i}{2\pi\hbar}\right] \frac{dp_N}{2\pi\hbar} \cdot e^{\frac{i}{\hbar}\sum_{j=1}^{N} p_j (q_j - q_{j-1})} $$

$$ \cdot \left[1 - \frac{i}{\hbar}\delta t H(p_N, q_N)\right] \cdots \left[1 - \frac{i}{\hbar}\delta t H(p_1, q_1)\right] $$

with $q_N = q$, $q_0 = q'$.

Now since

$$\lim_{N\to\infty} \prod_{i=1}^{N} \left[1 - \frac{i}{\hbar} H(p_i,q_i)\frac{t-t'}{N}\right] = \lim_{N\to\infty} e^{-\frac{i}{\hbar}\frac{t-t'}{N}\sum_{i=1}^{N} H(p_i,q_i)}$$

$$\Rightarrow U(q,t;q',t') = \lim_{N\to\infty} \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N-1} \frac{dq_i\, dp_i}{2\pi\hbar}\right] \frac{dp_N}{2\pi\hbar} \cdot$$

$$\cdot \exp\left\{ \frac{i}{\hbar} \delta t \sum_{j=1}^{N} \left[p_j \frac{q_j - q_{j-1}}{\delta t} - H(p_j, q_j)\right]\right\}$$

We denote this object by

$$U(q,t;q',t') = \int \left[\mathcal{D}q\ \mathcal{D}p\right] e^{\frac{i}{\hbar}\int_{t'}^{t} dt'' \left[p(t'')\dot{q}(t'') - H(p(t''),q(t''))\right]}$$

path integral ~ sum over all paths the particle could concievably travel.

$q(t) = q$, $q(t') = q'$.

For $H = \frac{p^2}{2m} + V(q)$ we can integrate over

p_j's : $$\int_{-\infty}^{\infty} dp_j\ e^{\frac{i}{\hbar}\left[p_j \frac{q_j - q_{j-1}}{\delta t} - \frac{p_j^2}{2m}\right]\delta t}$$

Use $$\int_{-\infty}^{\infty} dp\ e^{-\alpha p^2 + \beta p} = \sqrt{\frac{\pi}{\alpha}}\ e^{\beta^2/4\alpha}$$ (to do this correctly can rotate integration contour by 45° in the complex plane)

$$\Rightarrow \int_{-\infty}^{\infty} dp_j \; e^{\frac{i}{\hbar}\left[p_j \frac{q_j - q_{j-1}}{\delta t} - \frac{p_j^2}{2m}\right]\delta t} = \sqrt{\frac{2m\pi\hbar}{i\,\delta t}} \cdot$$

$$\cdot e^{\frac{m\hbar}{i}\cdot(-)\frac{1}{\hbar^2}\left(\frac{q_j - q_{j-1}}{\delta t}\right)^2 \cdot \frac{\delta t}{2}} = \sqrt{\frac{2\pi m\hbar}{i\,\delta t}} \; e^{\frac{i}{\hbar}\delta t \cdot \frac{m}{2}\cdot\left(\frac{q_j - q_{j-1}}{\delta t}\right)^2}$$

$$\Rightarrow U(q,t;q',t') = \lim_{N\to\infty} \int_{-\infty}^{\infty} \prod_{i=1}^{N-1} \frac{dq_i}{2\pi\hbar} \left[\frac{2\pi m\hbar}{i\,\delta t}\right]^{N/2} \cdot \frac{1}{2\pi\hbar} \cdot$$

$$\cdot \exp\left\{ \frac{i}{\hbar}\sum_{j=1}^{N} \delta t \underbrace{\left[\frac{m}{2}\left(\frac{q_j - q_{j-1}}{\delta t}\right)^2 - V(q_j)\right]}_{L(q_j,\,\dot{q}_j)}\right\} \Rightarrow$$

$$U = \lim_{N\to\infty} \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N-1} dq_i\right] \left[\frac{m}{2\pi\hbar i\,\delta t}\right]^{N/2} \cdot e^{\frac{i}{\hbar}\delta t \sum_{j=1}^{N} L(q_j,\,\dot{q}_j)} .$$

We denote this object by

$$U(q,t;q',t') = \mathcal{N}\int [\mathcal{D}q] \; e^{\frac{i}{\hbar}\int_{t'}^{t} dt'' \, L(q(t''),\,\dot{q}(t''))}$$

$$= \mathcal{N}\underbrace{\int [\mathcal{D}q]}_{\text{functional integral}} \; e^{\underbrace{\frac{i}{\hbar} S(q,t;q',t')}_{\text{functional of } q(t).}}$$

$\mathcal{N}$ ~ overal normalization factor, may be ∞.

Example | Free particle: $V(q)=0$.

$$U(q,t;q',t') = \lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar\delta t}\right]^{N/2}\int_{-\infty}^{\infty}\left[\prod_{i=1}^{N-1} dq_i\right] e^{\frac{i}{\hbar}\delta t\cdot\frac{m}{2}\sum_{j=1}^{N}\frac{(q_j-q_{j-1})^2}{\delta t^2}}$$

First integrate over q_1: need to do this integral:

$$I_1=\int_{-\infty}^{+\infty} dq_1\sqrt{\frac{m}{2\pi i\hbar\delta t}}\cdot e^{\frac{i}{2\hbar}\frac{m}{\delta t}\left[(q_1-q_0)^2+(q_2-q_1)^2\right]}.$$

$$[\dots] = 2q_1^2 - 2q_1(q_0+q_2)+q_0^2+q_2^2 = 2\left(q_1-\frac{q_0+q_2}{2}\right)^2+$$

$$+q_0^2+q_2^2-\frac{(q_0+q_2)^2}{2} = 2\underbrace{\left(q_1-\frac{q_0+q_2}{2}\right)}_{\tilde{q}_1}^2+\frac{1}{2}(q_0-q_2)^2.$$

$$I_1 = \sqrt{\frac{m}{2\pi i\hbar\delta t}}\; e^{\frac{i}{\hbar}\frac{m}{4\delta t}(q_0-q_2)^2}\cdot\int_{-\infty}^{\infty}d\tilde{q}_1\; e^{\frac{i}{2\hbar}\frac{m}{\delta t}\cdot 2\tilde{q}_1^2}$$

$$= \sqrt{\frac{\cancel{m}}{2\cancel{\pi}i\cancel{\hbar}\cancel{\delta t}}}\; e^{\frac{i}{\hbar}\frac{m}{4\delta t}(q_0-q_2)^2}\cdot\sqrt{\frac{\cancel{\pi}\cancel{\hbar}\cancel{\delta t}}{-\cancel{m}i}} = \frac{1}{\sqrt{2}}e^{\frac{i}{\hbar}\frac{m}{2\delta t}(q_0-q_2)^2}$$

$$\Rightarrow U(q,t;q',t') = \sqrt{\frac{m}{2\pi i\hbar\, 2\delta t}}\lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar\delta t}\right]^{\frac{N-2}{2}}\int_{-\infty}^{\infty}\left[\prod_{i=2}^{N-1}dq_i\right]\cdot$$

$$\cdot\, e^{\frac{i}{\hbar}\delta t\frac{m}{2}\left[\frac{1}{2}\left(\frac{q_0-q_2}{\delta t}\right)^2+\sum_{j=3}^{N}\left(\frac{q_j-q_{j-1}}{\delta t}\right)^2\right]}$$

$\Rightarrow$ note new $1/2$ factors $\Rightarrow$ iterate $\Rightarrow$ in the end one gets

$$U_{\text{free}}(q,t;q',t') = \sqrt{\frac{m}{2\pi i\hbar N\delta t}} \cdot \exp\left\{ \frac{i}{\hbar} \frac{1}{N\delta t} \cdot \frac{m}{2}(q-q')^2 \right\}$$

$\Rightarrow$ as $N\delta t = t - t'$ $\Rightarrow$

$$\boxed{U_{\text{free}}(q,t;q',t') = \sqrt{\frac{m}{2\pi i\hbar(t-t')}} \cdot \exp\left\{ \frac{i}{\hbar} \cdot \frac{m}{2} \cdot \frac{(q-q')^2}{t-t'} \right\}}$$

Feynman kernel for a free particle.

& one can also do the integral for harmonic oscillator, $V(q) = \frac{m}{2}\omega^2 q^2$.

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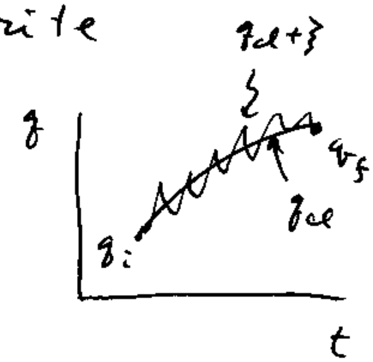
as $U(q,t;q',t') = \mathcal{N}\int[\mathcal{D}q]\, e^{\frac{i}{\hbar} S(q,t;q',t')}$ $\Rightarrow$

$\Rightarrow$ when $\hbar \to 0$ recover classical physics, have to minimize the action S. $\Rightarrow$ get classical EOM.

<u>Quasi-classical approximation</u>: write

$$q(t) = \underbrace{q_{cl}(t)}_{\text{classical trajectory}} + \underbrace{\xi(t)}_{\text{quantum fluctuations}}$$



Demand that $q_{cl}(t_i) = q_i$, $q_{cl}(t_f) = q_f$

$$\xi(t_i) = \xi(t_f) = 0$$