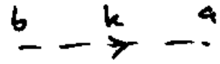
Last time | **Feynman Rules for QCD**

Quark Propagator 

$\frac{i}{\not p - m}\,\delta_{ij}$

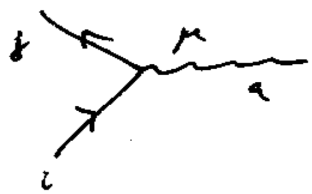
Ghost Propagator 

$\frac{i}{k^2+i\varepsilon}\,\delta_{ab}$

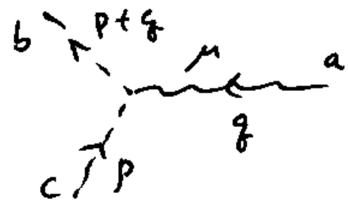
Gluon Propagator 

$\frac{-i}{k^2+i\varepsilon}\,\delta_{ab}\left[g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$

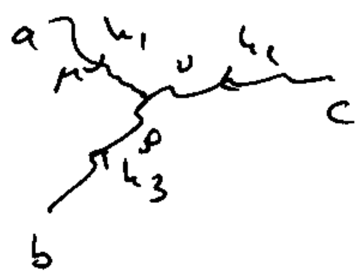
(Lorenz gauge)

Quark - Gluon Vertex 

$ig\,\gamma^\mu\,(T^a)_{ji}$

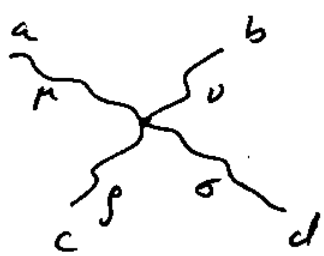
Ghost - Gluon Vertex 

$g\,f^{abc}(p+q)^\mu$

3 - Gluon Vertex 

$g\,f^{abc}\left[(k_1-k_3)_\nu\, g_{\mu\rho} + \right.$

$\left. + \text{perm's}\right]$

4 - Gluon Vertex 

$-ig^2\left[f^{abe}f^{cde}\cdot\right.$

$\cdot(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})$

$\left.+\text{perm's}\right]$

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LC gauge $\quad n\cdot A = A^+ = 0 \Rightarrow$ gluon prop. is $\frac{-i}{k^2+i\varepsilon}\,\delta^{ab}\left[g_{\mu\nu}-\frac{k_\mu n_\nu + k_\nu n_\mu}{n\cdot k}\right]$

& there is no ghost field

# Renormalization of QCD (cont'd)

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{0\mu\nu} F_0^{a\mu\nu} - \frac{1}{2\xi_0}\left(\partial_\mu A_0^{a\mu}\right)^2 + \left(\partial_\mu \bar{\eta}^a_0\right) D^\mu \eta^a_0$$
$$+ \bar{\Psi}_0 [i\not{D} - m_0] \psi_0$$

$\Rightarrow$ $\psi = \frac{1}{\sqrt{Z_2}}\psi_0$, $A_\mu = \frac{1}{\sqrt{Z_3}} A_{0\mu}$, $\eta^a = \frac{1}{\sqrt{Z_2^\eta}}\eta_0^a$

$\Rightarrow$ got a long Lagrangian written in terms of renormalized fields.

We write:

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + (\partial_\mu \bar{\eta}^a) D^\mu \eta^a +$$

$$+ \bar{\psi}[i\partial\!\!\!/ - m]\psi + \bar{\psi}[i\partial\!\!\!/ \delta_2 - \delta_m]\psi +$$

$$+ g\delta_1 \bar{\psi} A\!\!\!/ \psi + \delta_2^\eta \partial_\mu \bar{\eta}^a \partial^\mu \eta^a + g\,\delta_1^\eta \partial^\mu \bar{\eta}^a f^{abc} A^b_\mu \eta^c$$

$$- \frac{1}{4}\delta_3 (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - g\,\delta_1^{3g} f^{abc}(\partial_\mu A^a_\nu) A^{b}_\mu A^c_\nu -$$

$$- \frac{g^2}{4} \delta_1^{4g} f^{eab} A^a_\mu A^b_\nu f^{ecd} A^c_\mu A^d_\nu$$

where

$$\delta_2 = Z_2 - 1, \quad \delta_3 = Z_3 - 1, \quad \delta_2^\eta = Z_2^\eta - 1,$$

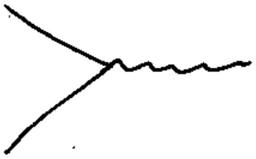
$$\delta_m = Z_2 m_0 - m, \quad \delta_1 = \frac{g_0}{g} Z_2 Z_3^{1/2} - 1 = Z_1 - 1$$

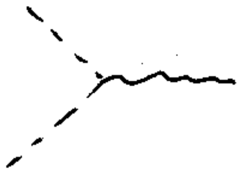
$$\delta_1^\eta = \underbrace{\frac{g_0}{g} Z_2^\eta Z_3^{1/2}}_{= Z_1^\eta} - 1, \quad \delta_1^{3g} = \frac{g_0}{g} Z_3^{3/2} - 1,$$

$$\delta_1^{4g} = \frac{g_0^2}{g^2}(Z_3)^2 - 1, \quad \xi = \frac{\xi_0}{Z_3}$$

8 counterterms! However, there are relations between the counterterm coefficients, as there are 5 indep. parameters: $Z_2, Z_3, Z_2^\eta, g_0, \underbrace{m_0}_{\text{or } Z_1,\, \delta_m}$.

(Slavnov-Taylor identities: like Ward identities in QED which had $Z_1 = Z_2$, but more subtle) at <u>one-loop</u> level (all $\delta$'s are order-$g^2$):

(a)  $g Z_1 = g_0 Z_2 Z_3^{1/2}$

(b)  $g Z_1^{\gamma} = g_0 Z_2^{\gamma} Z_3^{1/2}$

(c)  $g Z_1^{3g} = g_0 Z_3^{3/2}$

(d)  $g^2 Z_1^{4g} = g_0^2 (Z_3)^2$

$Z_1 = 1 + \delta_1$

$Z_1^{\gamma} = 1 + \delta_1^{\gamma}$

$Z_1^{3g} = 1 + \delta_1^{3g}$

$Z_1^{4g} = 1 + \delta_1^{4g}$

$\Rightarrow$ expand: (a) $g_0 = g \dfrac{Z_1}{Z_2 Z_3^{1/2}} = g\left[1 + \delta_1 - \delta_2 - \frac{1}{2}\delta_3 + \dots\right]$

(b) $g_0 = g \dfrac{Z_1^{\gamma}}{Z_2^{\gamma} Z_3^{1/2}} = g\left[1 + \delta_1^{\gamma} - \delta_2^{\gamma} - \frac{1}{2}\delta_3 + \dots\right]$

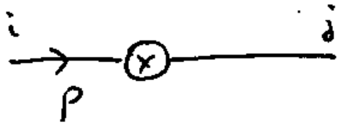
(c) $g_0 = g \dfrac{Z_1^{3g}}{Z_3^{3/2}} = g\left[1 + \delta_1^{3g} - \frac{3}{2}\delta_3 + \dots\right]$

(d) $g_0 = g \dfrac{\sqrt{Z_1^{4g}}}{Z_3} = g\left[1 + \frac{1}{2}\delta_1^{4g} - \delta_3 + \dots\right]$

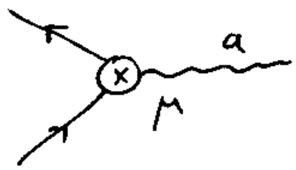
$\Rightarrow$ the relation between $g_0$ and $g$ should be the same $\Rightarrow$ $\boxed{\delta_1 - \delta_2 = \delta_1^{\gamma} - \delta_2^{\gamma} = \delta_1^{3g} - \delta_3 = \frac{1}{2}(\delta_1^{4g} - \delta_3).}$

Additional Feynman rules due to counterterms:

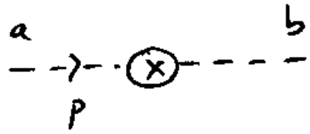

$$= -i\,\delta^{ab}\,\delta_3\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)$$



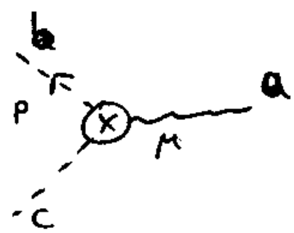

$$= i\left(\delta_2 \not p - \delta_m\right)\delta_{ij}$$ ~ color indeces



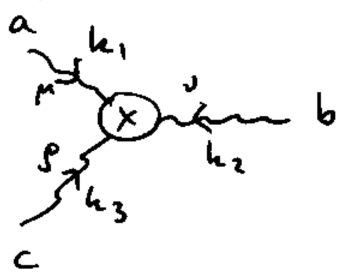

$$= i g\, T^a \gamma^\mu \delta_1$$



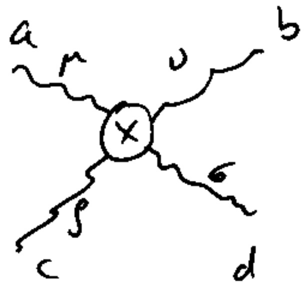

$$= i\, p^2\, \delta_2^{\eta}\, \delta^{ab}$$




$$= g\, p_\mu\, f^{abc}\, \delta_1^{\eta}$$




$$= -\delta_1^{3g}\cdot g\, f^{abc}\Big[(k_1-k_3)_\nu g_{\mu\rho} + (k_2-k_1)_\rho g_{\mu\nu} + (k_3-k_2)_\mu g_{\nu\rho}\Big]$$




$$= -i\,\delta_1^{4g} g^2\Big[f^{abe} f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ace} f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ade} f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\Big]$$

$\Rightarrow$ if we need to find the beta-function, use dim. reg. approach $\Rightarrow g \to g\mu^{\varepsilon/2}$, $\varepsilon = 4-d$

(using (a))

$$\Rightarrow g_0 = g\mu^{\varepsilon/2}\left[1+\delta_1-\delta_2-\tfrac{1}{2}\delta_3\right] \Rightarrow$$

$$\Rightarrow \underbrace{\frac{g_0^2}{4\pi}}_{\alpha_0} = \underbrace{\frac{g^2}{4\pi}}_{\alpha_\mu}\mu^{\varepsilon}\left[1+2(\delta_1-\delta_2)-\delta_3\right]$$

$$\Rightarrow \text{as } \beta(\alpha) = \mu^2\frac{d\alpha_\mu}{d\mu^2} \Rightarrow$$

$$\Rightarrow 0 = \mu^2\frac{d\alpha_0}{d\mu^2} = \mu^2\frac{d}{d\mu^2}\left[\alpha\mu^{\varepsilon}\left(1+2(\delta_1-\delta_2)-\delta_3\right)\right] =$$

$$= \beta(\alpha)\mu^{\varepsilon}\left(1+2(\delta_1-\delta_2)-\delta_3\right) + \frac{\varepsilon}{2}\alpha\mu^{\varepsilon}\left[1+2(\delta_1-\delta_2)-\delta_3\right]$$

$$+ \alpha\mu^{\varepsilon}\,\mu^2\frac{d}{d\mu^2}\left[2(\delta_1-\delta_2)-\delta_3\right]$$

Write $\delta_i = \alpha\cdot\bar{\delta}_i \Rightarrow 0 = \beta(\alpha)\left[1+2(\delta_1-\delta_2)-\delta_3\right]$

$$+ \beta(\alpha)\left[2(\delta_1-\delta_2)-\delta_3\right] + \frac{\varepsilon}{2}\alpha\left[1+2(\delta_1-\delta_2)-\delta_3\right]$$

$$\Rightarrow \beta(\alpha) = -\frac{\varepsilon}{2}\alpha\left[1-2(\delta_1-\delta_2)+\delta_3\right]$$

$\Rightarrow$ taking $\varepsilon \to 0$ limit yields:

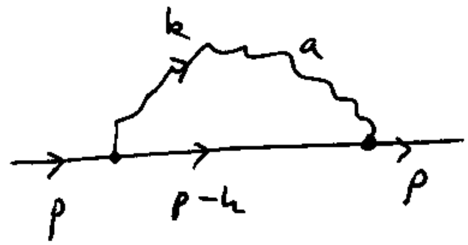
$$\boxed{\beta(\alpha) = \alpha\cdot\lim_{\varepsilon\to 0}\left\{\varepsilon\left[\delta_1-\delta_2-\tfrac{1}{2}\delta_3\right]\right\}}$$

# Running of QCD Coupling and Asymptotic Freedom.

We need to find $\delta_1$, $\delta_2$ and $\delta_3$. Start with $\delta_2$:

**Quark self-energy:**

$C_F = \frac{N_c^2-1}{2N_c} = T^a T^a$ fundamental Casimir



$$\int \frac{d^dk}{(2\pi)^d} (ig)^2 \underbrace{T^a T^a}_{C_F} \frac{-i}{k^2+i\varepsilon} \cdot \gamma^\mu \frac{i}{\not p - \not k} \gamma_\mu$$

dim. reg.

(for simplicity assume $m=0$ for quarks) $\Rightarrow$

$$\Rightarrow \text{ get } \quad -g^2 C_F \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+i\varepsilon} \gamma^\mu \frac{1}{\not p - \not k} \gamma_\mu = -i \Sigma_2(p)$$

$\Rightarrow$ adding the counterterm —⊗— get

$$\Sigma(p) = \Sigma_2(p) - \not p \, \delta_2 + \delta_m \quad \Rightarrow \text{ need finite } \Sigma$$

$$\Rightarrow \Sigma_2 - \not p \, \delta_2 = -i g^2 C_F \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+i\varepsilon} \frac{\overbrace{\gamma^\mu (\not p - \not k)\gamma_\mu}^{(2-d)(\not p - \not k)}}{(p-k)^2+i\varepsilon} - \not p \, \delta_2$$

$$= -i g^2 C_F (2-d) \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\not p - \not k}{\big[\underbrace{(1-x)k^2 + x\,(p-k)^2}_{k^2 - 2x p\cdot k + x p^2 = (k-xp)^2 + x(1-x)p^2}\big]^2} - \not p \, \delta_2$$

$$= -i g^2 C_F (2-d) \not p \int_0^1 dx \, (1-x) \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 + x(1-x)p^2]^2} - \not p \, \delta_2$$

$$= -ig^2 C_F (2-d)\, \not{p}\, \underset{\text{Wick rotation}}{i} \int_0^1 \frac{dx(1-x)}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{[-x(1-x)p^2]^{2-\frac{d}{2}}} \overset{-\not{p}\,\delta_2}{=} \Big|\ \varepsilon = 4-d$$

(we need only the singular term)

$$= \frac{g^2 C_F}{(4\pi)^2} (-2)\, \not{p}\, \underbrace{\Gamma\left(\frac{\varepsilon}{2}\right)}_{\frac{2}{\varepsilon}} \underbrace{\int_0^1 dx(1-x)}_{1/2} - \not{p}\,\delta_2 =$$

$$= \frac{-1}{(4\pi)^2} \frac{2g^2 C_F}{\varepsilon} \not{p} - \not{p}\,\delta_2 = \text{finite}$$

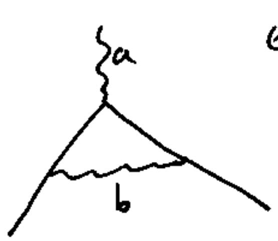
$$\Rightarrow \quad \delta_2 = -\frac{\alpha C_F}{2\pi\varepsilon}$$

~ different from QED only by color factor $C_F$.

To find $\delta_1$ need to find quark-gluon vertex corrections.

$= \text{finite}$

$ig\,\delta_1 \gamma^\mu T^a$



remember in QED have $\delta_1 = \delta_2 \Rightarrow$ the contribution of this graph to $\delta_1$ is

$$\underbrace{-\frac{\alpha}{2\pi\varepsilon}}_{\text{like in QED}} \otimes \begin{pmatrix}\text{color}\\ \text{factor}\end{pmatrix}$$

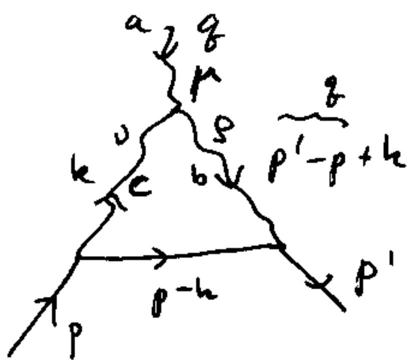
$$'' \quad T^b T^a T^b = T^b [T^a, T^b] + T^b T^b T^a$$

$$= i f^{abc} T^b T^c + C_F T^a$$

$$= i f^{abc} \underbrace{\frac{1}{2}[T^b, T^c]}_{\text{anti-symmetrized}} + C_F T^a = \frac{i^2}{2} \underbrace{f^{abc} f^{bcd}}_{N_c \delta^{ad}} T^d$$

$$+ C_F T^a = -\frac{1}{2} N_c T^d + C_F T^a = -\frac{1}{2N_c} T^a$$

$$\Rightarrow \text{get } \left(-\frac{\alpha}{2\pi\varepsilon}\right)\cdot\left(\frac{-1}{2N_c}\right) = \frac{\alpha}{4\pi N_c \varepsilon}.$$



$$= \int \frac{d^d k}{(2\pi)^d} \cdot (ig)^2 (-g)\, (\text{color factor}) \cdot \gamma^\rho \frac{i}{\not p - \not k} \gamma^\nu$$

$$\cdot \frac{-i}{k^2 + i\varepsilon} \frac{-i}{(p'-p+k)^2 + i\varepsilon} \cdot \Big[ (q-k)_\rho\, g_{\mu\nu} +$$

$p + q = p'$

$$+ (\underbrace{-p'+p}_{-q} - k - q)_\nu\, g_{\mu\rho} + (2k+q)_\mu\, g_{\nu\rho} \Big]$$

$$\text{color factor} = f^{abc}\, T^b T^c = f^{abc} \frac{1}{2}[T^b, T^c] =$$

$$= f^{abc} \frac{i}{2} f^{bcd}\, T^d = \frac{i}{2} N_c T^a$$

$\Rightarrow$ to find UV-divergent part of the graph assume that $k^\mu \gg p^\mu, p'^\mu, q^\mu$ $\Rightarrow$ one gets

$$\int \frac{d^d k}{(2\pi)^d} (-g^3) \left(\frac{-1}{2} N_c\right) T^a \frac{1}{(k^2)^3}\, \gamma^\rho (-\not k) \gamma^\nu \Big[ -k_\rho\, g_{\mu\nu}$$

$$- k_\nu\, g_{\mu\rho} + 2 k_\mu\, g_{\nu\rho} \Big] = \frac{-g^3}{2} N_c T^a \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^3}.$$

$$\cdot \left[ -k^2 \gamma_\mu \cdot 2 + \underbrace{2 k_\mu (-2) \not{k}}_{-4 \underbrace{k_\mu k_\nu}_{\frac{1}{d} k^2 g_{\mu\nu},\ d \to 4} \gamma^\nu} \right] = -\frac{g^3}{2} N_c T^a \cdot \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2}$$

$$\cdot \left[ -2\gamma_\mu - \gamma_\mu \right] = \frac{3}{2} g^3 N_c T^a \gamma_\mu \underbrace{\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2}}_{\frac{i}{(4\pi)^{d/2}} \underbrace{\Gamma\left(2 - \frac{d}{2}\right)}_{2/\varepsilon}} =$$

$$= \frac{3}{\varepsilon} g^3 N_c T^a \gamma_\mu \frac{i}{(4\pi)^2} =$$

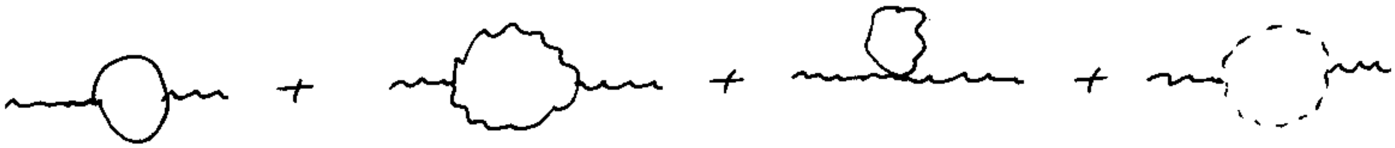
$$= i g \gamma_\mu T^a \left( \frac{3}{\varepsilon} \frac{\alpha}{4\pi} N_c \right)$$

$\Rightarrow$ contribution to $\delta_1$ is $-\frac{3}{\varepsilon} \frac{\alpha}{4\pi} N_c$

$$\Rightarrow \delta_1 = \frac{\alpha}{4\pi N_c \varepsilon} - \frac{3}{\varepsilon} \frac{\alpha}{4\pi} N_c = -\frac{\alpha}{4\pi} \frac{1}{\varepsilon} \left[ 3N_c - \frac{1}{N_c} \right]$$

$$\Rightarrow \delta_1 = -\frac{\alpha}{4\pi} \frac{1}{\varepsilon} \left[ 3N_c - \frac{1}{N_c} \right]$$

Finally, we also need $\delta_3$. To find it we calculate gluon self-energy up to $o(\alpha)$:



$+$  $=$ finite

Start with quark loop:


$$= \begin{pmatrix} \text{same as in} \\ \text{QED} \end{pmatrix} \otimes \begin{pmatrix} \text{color} \\ \text{factor} \end{pmatrix} \otimes N_f$$

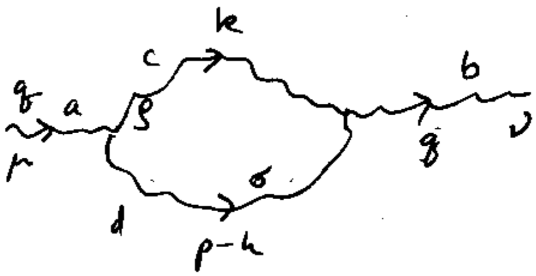
\# of quark flavors

$$\begin{pmatrix} \text{same as in} \\ \text{QED} \end{pmatrix} = -\frac{\alpha}{3\pi}\frac{2}{\varepsilon}, \qquad \begin{pmatrix} \text{color} \\ \text{factor} \end{pmatrix} = tr\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}$$

quarks

$$\Rightarrow \boxed{\delta_3^f = -\frac{\alpha}{3\pi}\frac{1}{\varepsilon} N_f}$$

Gluon loop:




$$= g^2 \cdot \underbrace{f^{acd} f^{cbd}}_{-N_c \delta^{ab}} \cdot \int \frac{d^dk}{(2\pi)^d} \frac{-i}{k^2} \frac{-i}{(q-k)^2}$$

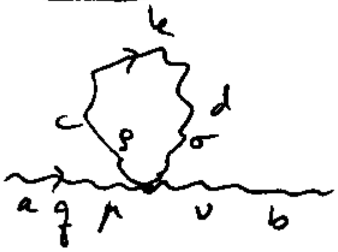
$$\cdot \left[ (2q-k)_\rho g_{\mu\sigma} + (-k-q)_\sigma g_{\mu\rho} + (2k-q)_\mu g_{\rho\sigma} \right] \cdot$$

$$\cdot \left[ (2k-q)_\nu g_{\rho\sigma} + (-q-k)_\sigma g_{\rho\nu} + (2q-k)_\rho g_{\nu\sigma} \right] \cdot \frac{1}{2}$$

symmetry factor

$$= \frac{g^2 N_c}{2} \delta^{ab} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} \cdot \frac{1}{(q-k)^2} \left[ \dots \right]\left[ \dots \right]$$

$$= -i\,\frac{\alpha N_c}{4\pi}\,\delta^{ab}\,\frac{1}{\varepsilon}\left[-q^2 g_{\mu\nu} - 2 q_\mu q_\nu\right]\cdot\frac{1}{6}$$




$$= \underbrace{\frac{1}{2}}_{\text{Symmetry factor}} \int \frac{d^dk}{(2\pi)^d}\,\frac{-i g_{\rho\sigma}}{k^2+i\varepsilon}\,\delta_{cd}\,(-ig^2)\Big[\cancel{f^{abe} f^{cde}}_{\,0}\cdot$$

$$\cdot\left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) + f^{ace} f^{bde}\left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}\right)$$

$$+ f^{ade} f^{bce}\left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}\right)\Big] = \frac{-g^2}{2} N_c \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}$$

$$\delta^{ab}\cdot g_{\rho\sigma}\left[\,2 g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}\right] = -\frac{g^2 N_c}{2}\,\delta^{ab}\cdot$$

$$\cdot\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}\cdot\left[2d\, g^{\mu\nu} - 2 g^{\mu\nu}\right] = -g^2 N_c\,\delta^{ab}\,(d-1)\, g^{\mu\nu}\cdot$$

$$\cdot\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}\,\frac{(q-k)^2}{(q-k)^2} = -g^2 N_c\,\delta^{ab}\,(d-1)\, g^{\mu\nu}\cdot \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\cdot$$

$$\cdot\frac{1}{\left[(k-xq)^2 + x(1-x)q^2\right]^2}\cdot\left(q^2 - 2q\cdot k + k^2\right) = -g^2 N_c\,\delta^{ab}\,(d-1)\, g^{\mu\nu}\cdot$$

$$\cdot\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{\left[k^2 + x(1-x)q^2\right]^2}\left(q^2 - 2x q^2 + k^2 + x^2 q^2\right) = -i g^2 N_c\,\delta^{ab}\cdot$$

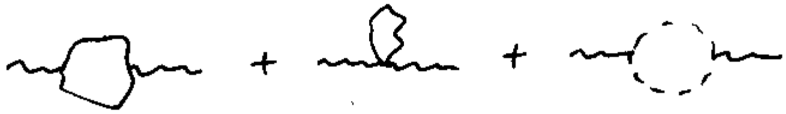
$$\cdot(d-1)\, g^{\mu\nu}\int_0^1 dx \int \frac{d^dk_E}{(2\pi)^d}\;\frac{-k_E^2 + q^2\left(1 - 2x + x^2\right)}{\left[-k_E^2 + x(1-x)q^2\right]^2} =$$

$$= -ig^2 N_c \delta^{ab}(d-1) g^{\mu\nu} \int_0^1 dx \cdot \frac{1}{(4\pi)^{d/2}} \left[-x(1-x)q^2\right]^{\frac{d}{2}-2} \cdot$$

$$\cdot \left\{ + \frac{d}{2}\Gamma\left(1-\frac{d}{2}\right) (\cancel{-})x(1-x)q^2 + q^2(1-x)^2 \Gamma\left(2-\frac{d}{2}\right)\right\} = \Big|_{d=4-\varepsilon}$$

$$= -i \frac{\alpha N_c}{4\pi}\cdot 3\, \delta^{ab} g^{\mu\nu} \left\{ -\frac{4}{\varepsilon}\cdot\frac{1}{6} q^2 + \frac{2}{\varepsilon} q^2 \frac{1}{3}\right\} = 0$$

$$\Rightarrow \int \frac{d^d k}{k^2} = 0 .$$

$\Rightarrow$  $= i\,\delta^{ab}\,\frac{1}{\varepsilon}\,\frac{\alpha N_c}{4\pi}\cdot\frac{10}{3}\cdot$

$$\cdot \left[q^2 g_{\mu\nu} - q_\mu q_\nu\right]$$

$\Rightarrow$ contrib. to $\delta_3$ is $\delta_3^g = \frac{1}{\varepsilon}\,\frac{\alpha N_c}{4\pi}\cdot\frac{10}{3}$

$$\Rightarrow \delta_3 = \delta_3^g + \delta_3^f = \frac{\alpha}{4\pi}\,\frac{1}{\varepsilon}\left[\frac{10}{3}N_c - \frac{4}{3}N_f\right]$$

$$\beta_{QCD}(\alpha) = \alpha \lim_{\varepsilon\to 0}\left\{\varepsilon\left[\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right]\right\} = \alpha\cdot\frac{\alpha}{4\pi}\cdot$$

$$\cdot\left\{-3N_c + \frac{1}{N_c} + \underbrace{2C_F}_{N_c - \frac{1}{N_c}} - \frac{5}{3}N_c + \frac{2}{3}N_f\right\} = \frac{\alpha^2}{4\pi}\cdot\left\{-\frac{11}{3}N_c + \frac{2}{3}N_f\right\}$$

$$\Rightarrow \beta_{QCD}(\alpha) = -\frac{\alpha^2}{12\pi}\left[11 N_c - 2 N_f\right]$$