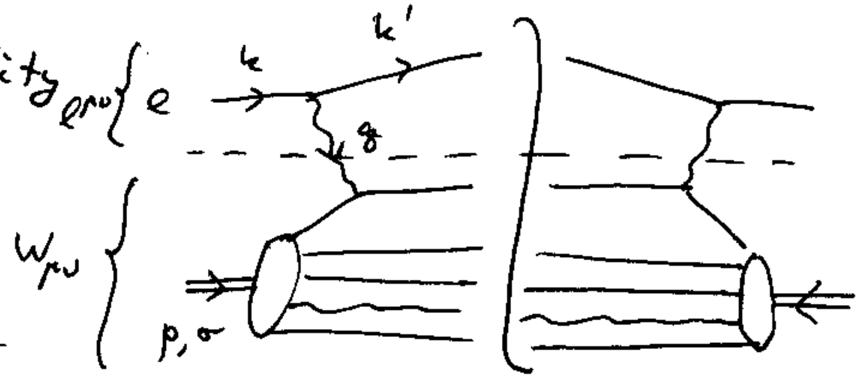
Last time | Parton Model and DIS (cont'd)

DIS (cont'd)

$$\boxed{\begin{aligned} Q^2 &= -q^2 \\ x &= \frac{Q^2}{2p\cdot q} \end{aligned}}$$ ~ photon virtuality; ~ Bjorken-x



$\hat{s} = (p+q)^2 \Rightarrow x = \dfrac{Q^2}{Q^2 + \hat{s}}$

(neglecting m_p) $\Rightarrow$ large $\hat{s}$ $\Leftrightarrow$ small-x

In proton's rest frame we wrote DIS cross-section as

$$\boxed{\frac{d\sigma}{d^3k'} = \frac{\alpha_{EM}^2}{Q^4 E E'}\, l_{\mu\nu} W^{\mu\nu}}$$

where $l_{\mu\nu} = 2\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}\, k\cdot k'\right)$

(leptonic tensor)

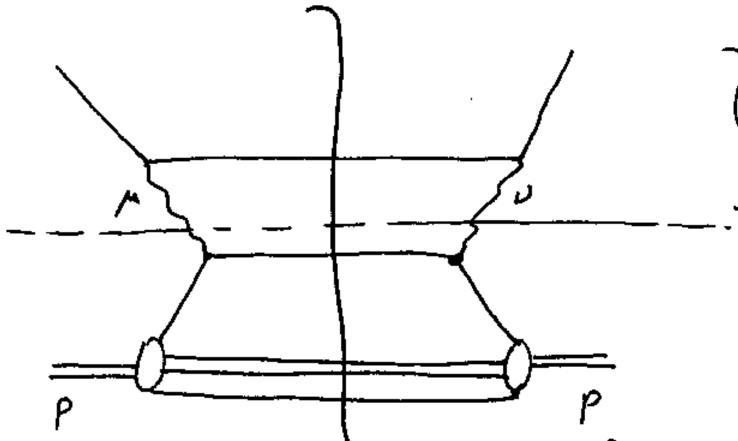
$$W_{\mu\nu} = \frac{1}{4\pi m}\int d^4x\, e^{iq\cdot x} \langle p | j_\mu(x)\, j_\nu(0) | p\rangle .$$

(hadronic tensor)

All QCD physics of DIS is in $W_{\mu\nu}$: in general it is non-perturbative and can not be calculated from first principles.

$$W_{\mu\nu} = \frac{1}{4\pi m} \int d^4x \, e^{iq\cdot x} \langle p | j_\mu(x) \, j_\nu(0) | p \rangle$$

(averaged over σ)



$\} \; l_{\mu\nu}$ (EM part)

$\} \; W_{\mu\nu}$ (hadronic part)

$$W_{\mu\nu}(p,q) = A(x,Q^2)\, p_\mu p_\nu + B(x,Q^2)\, q_\mu q_\nu + C(x,Q^2)\, g_{\mu\nu} +$$

$$+ D(x,Q^2)\,(p_\mu q_\nu + p_\nu q_\mu) + E(x,Q^2)\,(p_\mu q_\nu - p_\nu q_\mu) +$$

$$+ F(x,Q^2)\, \varepsilon_{\mu\nu\rho\sigma}\, p^\rho q^\sigma$$

$F = 0$ in $\gamma^* p$, $\gamma^* A$ (F comes from γ_5's, appears in ν DIS).

(1) $q_\mu W^{\mu\nu} = 0$ (current conservation)

$q_\mu \to \partial_\mu \Rightarrow \partial_\mu j^\mu(x) = 0$

$$A\, p_\nu\,(p\cdot q) + B\, q_\nu \cdot q^2 + C\, q_\nu + D\,(p\cdot q\, q_\nu + q^2 p_\nu)$$

$$+ E\,(p\cdot q\, q_\nu - q^2 p_\nu) = 0$$

(2) $q_\nu W^{\mu\nu} = 0 = A\, p_\mu\,(p\cdot q) + B\, q^2 q_\mu + C q_\mu + D(p\cdot q\, q_\mu +$

$+ q^2 p_\mu) + E\,(p_\mu q^2 - p\cdot q\, q_\mu) = 0$

$(1) - (2) = 0 \quad \Rightarrow \quad E = 0.$

p_μ and q_μ are independent $\Rightarrow$

$$0 = A\, p\cdot q + D q^2 \qquad 0 = B q^2 + C + D\, p\cdot q$$

$$D = -A \frac{p\cdot q}{q^2} \qquad B = -\frac{1}{q^2} C + A\left(\frac{p\cdot q}{q^2}\right)^2$$

$$W_{\mu\nu} = A\left[p_\mu p_\nu - \frac{p\cdot q}{q^2}(p_\mu q_\nu + p_\nu q_\mu) + \left(\frac{p\cdot q}{q^2}\right)^2 q_\mu q_\nu\right] + C\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right]$$

Usually one writes

$$W_{\mu\nu} = -W_1(x, Q^2)\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] + \frac{W_2(x, Q^2)}{m_p^2} \cdot \left[p_\mu p_\nu - \frac{p\cdot q}{q^2}(p_\mu q_\nu + p_\nu q_\mu) + \left(\frac{p\cdot q}{q^2}\right)^2 q_\mu q_\nu\right]$$

W_1 & W_2 are <u>structure</u> <u>functions</u> Def.

Using $q_\mu \ell^{\mu\nu} = q_\nu \ell^{\mu\nu} = 0$ yields

$$\ell_{\mu\nu} W^{\mu\nu} = -W_1 (-4\underbrace{k\cdot k'}_{2\varepsilon\varepsilon' \sin^2\frac{\theta}{2}}) + \frac{2W_2}{m_p^2}\underbrace{\left[2\, p\cdot k\; p\cdot k' - m^2 k\cdot k'\right]}_{2m^2\varepsilon\varepsilon' - 2m^2 \varepsilon\varepsilon' \sin^2\frac{\theta}{2}}$$

$$2m^2\varepsilon\varepsilon' - 2m^2 \varepsilon\varepsilon' \sin^2\frac{\theta}{2} = 2m^2 \varepsilon\varepsilon' \cos^2\frac{\theta}{2}$$

$$q_\mu l^{\mu\nu} = (k-k')_\mu \, 2\left(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu} k\cdot k'\right) =$$

$$= 2\left(k^2 k'^\nu + \cancel{k^\nu k\cdot k'} - \cancel{k^\nu k\cdot k'} - \cancel{k\cdot k\, k'^\nu} - k^\nu k'^2\right.$$

$$\left. + \cancel{k'^\nu k\cdot k'}\right) = 2\left(k^2 k'^\nu - k'^2 k^\nu\right) \simeq 0 \quad \text{as } k^2 \simeq k'^2 \simeq 0$$

(neglect electron's mass), $q_\nu l^{\mu\nu} = 0$ (similar)

$$\Rightarrow l_{\mu\nu} W^{\mu\nu} = l^{\mu\nu}\left[-W_1\left(g_{\mu\nu} - \cancelto{0}{\frac{q_\mu q_\nu}{q^2}}\right) + \frac{W_2}{m_p^2}\right.$$

$$\left.\left(p_\mu p_\nu - \frac{p\cdot q}{q^2}\left(\cancelto{0}{p_\mu q_\nu} + \cancelto{0}{p_\nu q_\mu}\right) + \left(\frac{p\cdot q}{q^2}\right)^2 \cancelto{0}{q_\mu q_\nu}\right)\right]$$

$$= -l^\mu{}_\mu W_1 + \frac{W_2}{m_p^2} p_\mu p_\nu l^{\mu\nu} = \left| \begin{array}{l} \text{as } l^{\mu\nu} = 2(k^\mu k'^\nu + \\ + k^\nu k'^\mu - g^{\mu\nu} k\cdot k') \end{array}\right.$$

$$= -2\left(2k\cdot k' - \overset{g^\mu_\mu}{4}\, k\cdot k'\right) W_1 + \frac{W_2}{m_p^2}\, 2\left(2p\cdot k\, p\cdot k' - p^2 k\cdot k'\right)$$

$$= 4\, k\cdot k'\, W_1 + 2\frac{W_2}{m_p^2}\left(2p\cdot k\, p\cdot k' - m_p^2\, k\cdot k'\right)$$

Remember: $k = (\varepsilon, 0, 0, \varepsilon)$, $k' = (\varepsilon', \varepsilon'\sin\theta, 0, \varepsilon'\cos\theta)$

$$p = (m_p, \vec{0})$$

$$\Rightarrow k\cdot k' = 2\varepsilon\varepsilon' \sin^2(\theta/2)\,; \quad p\cdot k = m_p \varepsilon\,, \; p\cdot k' = \varepsilon' m_p$$

$$\ell_{\mu\nu} W^{\mu\nu} = 4 \varepsilon \varepsilon' \left[2 W_1 \sin^2 \frac{\theta}{2} + W_2 \cos^2 \frac{\theta}{2} \right]$$

$$\frac{d\sigma}{d^3k'} = \frac{4 \alpha_{EM}^2}{Q^4} \left[2 W_1 \sin^2 \frac{\theta}{2} + W_2 \cos^2 \frac{\theta}{2} \right]$$

By varying the angle θ can separate W_1 & W_2 contributions in experiments.

Usually one defines $F_1(x, Q^2) = M_p W_1(x, Q^2)$, $F_2(x, Q^2) = \nu \, W_2(x, Q^2)$

The Parton Model.

Sterman ch 14, Peskin 17.5

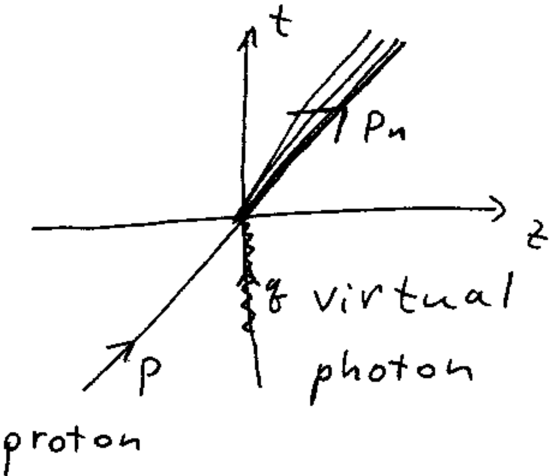
Go to Infinite Momentum Frame:

$$p_\mu \approx \left(p + \frac{m^2}{2p}, \; 0, \; 0, \; p \right)$$

$$\qquad\quad 0 \qquad\qquad 1 \quad 2 \quad 3$$

$$q_\mu = \left(q_0, \; \underset{\sim}{q}, \; 0 \right)$$

$$\qquad 0 \qquad 1,2 \qquad 3$$



Q^2 and x are 2 invariants

↖ large, $Q \gg \Lambda_{QCD}$

$$p \cdot q = m\nu = q_0 \cdot p$$

$$\Rightarrow q_0 = \frac{m\nu}{p} \sim \text{small as } p \text{ goes large}$$

$$\Rightarrow Q^2 = -q^2 = \underset{\sim}{q}^2$$

$$F_1(x, Q^2) = m_p W_1(x, Q^2)$$

$$F_2(x, Q^2) = \nu W_2(x, Q^2) = \frac{Q^2}{2m_p x} W_2(x, Q^2)$$

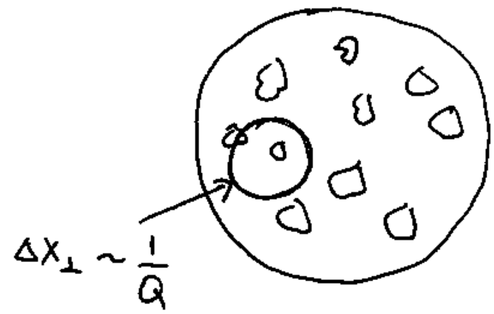
$$\boxed{\frac{d\sigma}{d^3k'} = \frac{4\alpha_{EM}^2}{Q^4}\left[2\frac{1}{m_p}F_1(x,Q^2)\sin^2\frac{\theta}{2} + \frac{2m_p x}{Q^2}F_2(x,Q^2)\cdot\cos^2(\theta/2)\right]}$$

$Q^2 = \not{q}^2 \Rightarrow$ photon acts like a <u>microscope</u>

in <u>transverse</u> plane:

$$\Delta x_\perp \cdot q_\perp \sim 1 \qquad (\hbar = 1)$$

$$\Delta x_\perp \sim \frac{1}{q_\perp} \sim \frac{1}{Q}$$

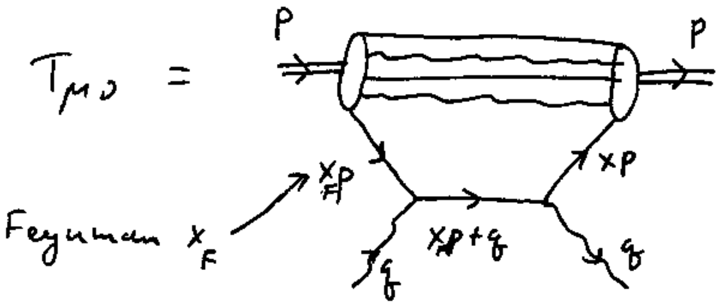


proton

large Q ~ resolve just 1 quark

Define
$$\boxed{T_{\mu\nu} = \frac{1}{4\pi m_p}\int d^4x\, e^{iq\cdot x}\,\frac{1}{2}\sum_\sigma \langle p,\sigma|\,T\,j_\mu(x)j_\nu(0)\,|p,\sigma\rangle}$$

$$W_{\mu\nu} = 2\,\mathrm{Im}\,(i\,T_{\mu\nu}) \qquad \text{(optical theorem)}$$

$T_{\mu\nu} =$ 

"Forward Amplitude"

Def Feynman-x: the fraction of proton's longitudinal momentum carried by struck quark

typical interaction time in proton's rest frame

is $\frac{1}{\Lambda_{QCD}} \Rightarrow$ boost to get $\frac{P}{m}\frac{1}{\Lambda} \equiv \tau_\Lambda$

int. time of DIS is $\tau_{DIS} \simeq \frac{1}{q^0}$, where

$q^0 \simeq \frac{Q^2}{2x_B p}$ $\left(= \frac{m\nu}{p} \leftarrow \text{as } m\nu = \frac{Q^2}{2x}\right)$ is struck quark's velocity: $\tau_{DIS} \simeq \frac{2x_B p}{Q^2}$.

time-ordered product: (denoted T)

$$\boxed{T\, j_\mu(x)\, j_\nu(y) \equiv \Theta(x^0-y^0)\, j_\mu(x) j_\nu(y) + \Theta(y^0-x^0)\, j_\nu(y) j_\mu(x)}$$

note: currents do not commute with each other in general $\Rightarrow$ not a trivial object.

$$2\,\mathrm{Im}\left(i\,T_{\mu\nu}\right) = 2\,\mathrm{Im}\Big[i \cdot \frac{4\pi^2 E_p}{m_p} \int d^4x\, e^{iq\cdot x} \langle p|\, \Theta(x^0)\, \overbrace{j_\mu(x)}^{e^{ip\cdot x} j_\mu(0) e^{-ip\cdot x}} \underset{\sum_n |n\rangle\langle n|}{\wedge} j_\nu(0) +$$

$$+ \Theta(-x^0) j_\nu(0) j_\mu(x)\, |p\rangle \Big] = 2 \cdot \frac{1}{4\pi m_p} \sum_n \mathrm{Re} \Big\{ \int d^4x\, e^{iq\cdot x + ip\cdot x - ip_n \cdot x}$$

$$\cdot \Theta(x^0) \langle p| j_\mu(0)|n\rangle\langle n| j_\nu(0)|p\rangle + \int d^4x\, e^{iq\cdot x + ip_n\cdot x - ip\cdot x}\, \Theta(-x^0) \cdot$$

$$\langle p| j_\nu(0)|n\rangle\langle n| j_\mu(0)|p\rangle \Big\} = \sum_n 2\, \frac{1}{4\pi m_p} \Big\{ (2\pi)^3\, \delta(\vec{q}+\vec{p}-\vec{p}_n) \cdot \mathrm{Re}$$

$$\left(\frac{-1}{i(q^0+p^0-p_n^0+i\varepsilon)} \langle p| j_\mu(0)|n\rangle\langle n| j_\nu(0)|p\rangle\right) + (2\pi)^3\, \delta(\vec{q}+\vec{p}_n-\vec{p}) \cdot$$

$$\cdot \mathrm{Re}\left(\frac{1}{i(q^0+p_n^0-p^0-i\varepsilon)} \langle p| j_\nu(0)|n\rangle\langle n| j_\mu(0)|p\rangle\right) \Big\}$$

$\propto \delta(q^0+p_n^0-p^0)$ $= 0$ not physical $\Rightarrow$ drop (after including $\delta(q^0+p_n^0-p^0)$)

$$= 2\, \frac{1}{4\pi m_p} \sum_n (2\pi)^3\, \delta(\vec{q}+\vec{p}-\vec{p}_n)\, (-)\left(\mathrm{Im}\, \frac{1}{q^0+p^0-p_n^0+i\varepsilon}\right) \cdot \langle p| j_\mu(0)|n\rangle$$

$$\langle n| j_\nu(0)|p\rangle = \left| \text{as } \mathrm{Im}\, \frac{1}{x+i\varepsilon} = -\pi\,\delta(x) \right. = \frac{1}{4\pi m_p} \sum_n (2\pi)^4\, \delta^4(q+p-p_n)$$

$$\cdot \langle p| j_\mu(0)|n\rangle\langle n| j_\nu(0)|p\rangle = W_{\mu\nu} \quad \text{as desired.}$$

if x is small ($\lesssim 1$) and Q is large

$\Rightarrow$ $\tau_{DIS} \ll \tau_\Lambda$ interaction is "instantaneous"

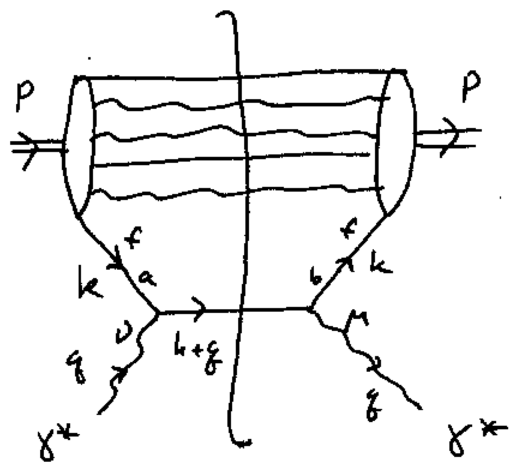
Define light cone variables:

for vector V^μ one has $V^+ = \frac{1}{\sqrt{2}}(V^0 + V^3)$

$\underline{V} = (V^1, V^2)$ $\qquad V^- = \frac{1}{\sqrt{2}}(V^0 - V^3)$

(2d transverse vector)

$$V_1 \cdot V_2 = V_{1\mu} V_2^{\,\mu} = V_{1+} V_{2-} + V_{1-} V_{2+} - \underline{V}_1 \cdot \underline{V}_2$$

$W_{\mu\nu} =$ 

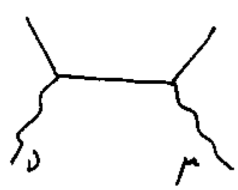
$\Big\}\ A^f_{ab}(p,k)$

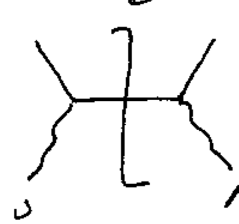
e.g.
$p_+ = \frac{p}{\sqrt{2}}$

$p_- = \frac{m^2}{2\sqrt{2}\,p}$

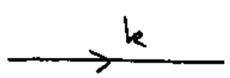
$p_+ \gg p_-$

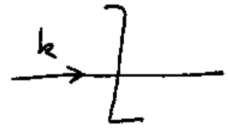
as $W_{\mu\nu} = 2\,\mathrm{Im}\,(i\,T_{\mu\nu})$

$T_{\mu\nu} =$ 

$W_{\mu\nu} =$ 

cut (Im part of a propagator)



$\frac{i}{k^2 - m^2 + i\varepsilon}$ $\Rightarrow$ 

$2\pi\,\delta^{(+)}(k^2 - m^2)$

as $2\,\mathrm{Im}\,\frac{-1}{k^2 - m^2 + i\varepsilon} = +2\pi\,\delta^{(+)}(k^2 - m^2)$

We write

$$W_{\mu\nu} = \frac{1}{\cancel{4\pi} 2 m} \sum_f e_f^2 \int d^4k \; A^f_{ab}(p,k) \left[\gamma_\mu \, \gamma\cdot(k+q) \, \gamma_\nu \right]_{ba}$$

$\cdot \cancel{(2\pi)} \, \delta((k+q)^2)$ where A^f_{ab} is the rest of the diagram (see p. 46).

Start calculating assuming that

$$Q^2 \gg k^2, \; \underline{k}\cdot\underline{q} \quad , \quad k_+ \gg k_- \quad (\text{IMF})$$

$$(k+q)^2 = k^2 + 2k_+ q_- + 2k_- q_+ - \underline{k}\cdot\underline{q} - Q^2$$

$q_3 = 0 \Rightarrow q_+ = q_- \Rightarrow$ as $k_+ \gg k_- \Rightarrow$ drop $2k_- q_+$

dropping $k^2, \underline{k}\cdot\underline{q} \ll Q^2$ get

$$(k+q)^2 \simeq 2k_+ q_- - Q^2$$

$$\Rightarrow \delta((k+q)^2) \simeq \delta(2k_+ q_- - Q^2) = \delta\left(\frac{k_+}{p_+} 2p_+ q_- - Q^2\right)$$

as $p\cdot q \simeq \cancel{2} p_+ q_- \Rightarrow$ and $x_{Bj} = \dfrac{Q^2}{2p\cdot q}$

$$\Rightarrow \delta((k+q)^2) \simeq \frac{x_{Bj}}{Q^2} \delta\left(x_{Bj} - \frac{k_+}{p_+}\right)$$

$$\Rightarrow \boxed{x_{Bj} = \frac{k_+}{p_+}}$$ Feynman x = Bjorken x

physical meaning: light cone momentum fraction of struck quark!

$$\gamma_0 (k+q) = \gamma_+ (k_- + q_-) + \gamma_- (k_+ + q_+) - \underline{\gamma} \cdot (\underline{k} + \underline{q})$$

after d^4k : $\gamma_+ \to p_+ \quad \gamma_- \to p_- \quad \underline{\gamma} \to \underline{p} = 0$

$\Rightarrow$ as $p_+ \gg p_-$ keep γ_+ only, $q_- \simeq \frac{Q^2}{x \cdot 2p_+}$, $(k_- \ll q_-)$

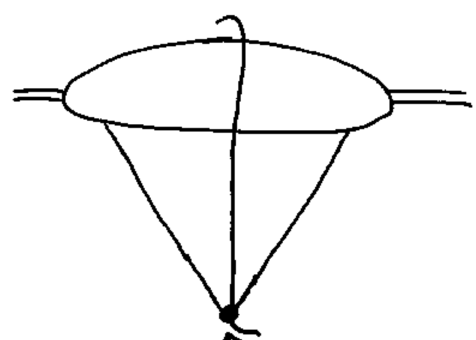
$$W_{\mu\nu} = \frac{1}{4 m p_+} \sum_f e_f^2 \int d^4k \; A^f_{ab}(p,k) \left[\gamma_\mu \gamma_+ \gamma_\nu \right]_{ba} \cdot \delta\left(x - \frac{k_+}{p_+} \right)$$

symmetrize, as $W_{\mu\nu}$ is symmetric

Concentrate on $W_{ij} \sim \frac{1}{2} \left[\gamma_i \gamma_+ \gamma_j + \gamma_j \gamma_+ \gamma_i \right] =$

$= -\frac{1}{2} \gamma_+ \{ \gamma_i, \gamma_j \} = -g_{ij} \gamma_+$ (we used $W_{ij} = W_{ji}$)

DIS now looks like



$\gamma_+ \delta\left(x - \frac{k_+}{p_+}\right)$

(Mueller vertex)

We have $W_{ij} \propto g_{ij}$ from diagram calculations.

On the other hand, since $\underline{p} = 0$

$$W_{ij} = -W_1 \left(g_{ij} - \frac{q_i q_j}{q^2} \right) + \frac{W_2}{m_p^2} q_i q_j \left(\frac{p \cdot q}{q^2} \right)^2 =$$

$$= -W_1 g_{ij} + \frac{q_i q_j}{q^2} \left[W_1 + \frac{W_2}{m_p^2} \frac{(p \cdot q)^2}{q^2} \right] \propto g_{ij}$$

$$\Rightarrow \quad W_1 + \frac{W_2}{m^2}\frac{(p\cdot q)^2}{q^2} = 0$$

as $\nu = \frac{p\cdot q}{m}$ and $x = \frac{Q^2}{2p\cdot q} = -\frac{q^2}{2p\cdot q}$

we write $\boxed{\nu W_2 = 2 m x W_1}$ Callan-Gross Relation '69

follows from spin-$\frac{1}{2}$ nature of quarks!

(would be different for particles with different spin); equivalently: $\boxed{F_2(x,Q^2) = 2x F_1(x,Q^2)}$

Exercise: show that Callan-Gross relation leads to $\frac{d\sigma}{d^3k'} \sim \left[1 + \left(1 - \frac{\nu}{\varepsilon}\right)^2\right] W_1$

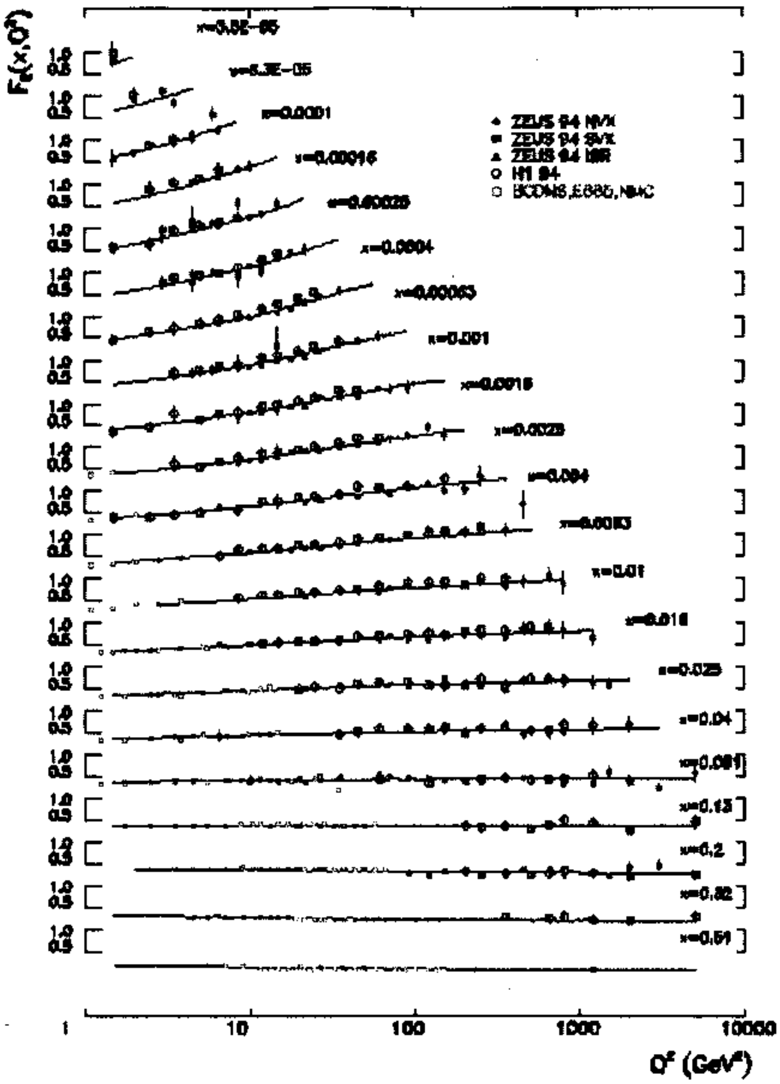
CG relation leads to

$$\nu W_2 = 2 m x W_1 = \cancel{2}\cancel{m} x \frac{1}{\cancel{2}\cancel{m} 2p_+} \sum_f e_f^2 \int d^4k \, A_{ab}^f(p,k) \cdot$$

$\cdot (\gamma_+)_{ba} \, \delta\left(x - \frac{k_+}{p_+}\right) \Rightarrow$ <u>defining</u> <u>quark distribution</u>:

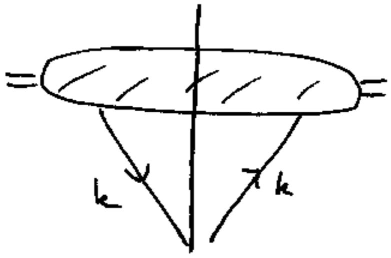
$$\boxed{q^f(x) \equiv \frac{1}{2p_+} \int d^4k \, A_{ab}^f(p,k) (\gamma_+)_{ba} \, \delta\left(x - \frac{k_+}{p_+}\right)}$$

we get $\boxed{\underset{(x,Q^2)}{\nu W_2} = \sum_f e_f^2 \, x q^f(x)}$ no Q^2-dependence only x-dependent <u>Bjorken scaling</u> (see attached)



Structure function $F_2(x, Q^2)$ plotted as a function of Q^2 for various values of x. Note that at large-x (lower curves) it is Q^2-independent; this is Bjorken scaling!

at low-x (upper curves) Bjorken scaling is violated.

$q^f(x) = \frac{1}{2p^+}$  $\Rightarrow$ often $p^f(x)$ is denoted $q(x)$.

$\gamma^+ \delta(x - \frac{k^+}{p^+})$

$\Rightarrow q^f(x, Q^2)$ counts # of quarks with light cone momentum x and transverse momentum $k_T \leq Q$.

parton distribution function $(q^f \sim a^+ a)$

$\Rightarrow$ for a free quark $A^f_{ab}(p,k)(\gamma^+)_{ba} = \delta^4(p-k)$.

plug in

$\underbrace{\bar{u}_b(p)(\gamma^+)_{ba} u_a(p)}_{= 2p^+} = 2p^+ \cdot \delta^4(p-k) \Rightarrow \boxed{q^f_{\text{quark}}(x) = \delta(x-1)}$

one quark at $x=1$