A Note on Spectral Representation

Consider a propagator in a scalar interacting theory (for simplicity we chose scalars):

$$D_F(p) = \int d^4x \, e^{ip\cdot x} \langle \Psi_0 | T \varphi(x) \varphi(0) | \Psi_0 \rangle .$$

Let's evaluate the Green function:

$$\langle \Psi_0 | T \varphi(x) \varphi(0) | \Psi_0 \rangle = \theta(x^0) \langle \Psi_0 | \varphi(x) \varphi(0) | \Psi_0 \rangle$$

$$+ \theta(-x^0) \langle \Psi_0 | \varphi(0) \varphi(x) | \Psi_0 \rangle = \left(\text{inserting a complete set of states} \right)$$

$$= \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 \, 2E_i} \Big[\theta(x^0) \langle \Psi_0 | \varphi(x) | n \rangle \langle n | \varphi(0) | \Psi_0 \rangle$$

$$+ \theta(-x^0) \langle \Psi_0 | \varphi(0) | n \rangle \langle n | \varphi(x) | \Psi_0 \rangle \Big]$$

(Here $|n\rangle$ is an n-particle state with momenta $p_1, p_2, \ldots, p_n$.)

Using Heisenberg picture, we write

$$\langle \Psi_0 | \varphi(x) | n \rangle = \langle \Psi_0 | e^{i\hat{p}\cdot x} \varphi(0) e^{-i\hat{p}\cdot x} | n \rangle = \langle \Psi_0 | \varphi(0) | n \rangle \cdot e^{-i x \cdot \sum_{j=1}^{n} p_j}$$

such that

$$\langle \Psi_0 | T \varphi(x) \varphi(0) | \Psi_0 \rangle = \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 \, 2E_i} \left| \langle \Psi_0 | \varphi(0) | n \rangle \right|^2$$

$$\cdot \left\{ \theta(x^0) e^{-i x^0 \sum_{j=1}^{n} p_j^0} + \theta(-x^0) e^{i x^0 \sum_{j=1}^{n} p_j^0} \right\} e^{i \vec{x} \cdot \sum_{k=1}^{n} \vec{p}_k}$$

where in the second term in curly brackets we redefined $\vec{p}_k \to -\vec{p}_k$ in the integrals.

Now, the expression in curly brackets is (see p.92 of class notes)

$$\theta(x^0)\, e^{-ix^0 \sum_{j=1}^{n} p_j^0} + \theta(-x^0)\, e^{ix^0 \sum_{j=1}^{n} p_j^0} = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, e^{-ik^0 x^0} \cdot \frac{i}{(k^0)^2 - \left(\sum_{j=1}^{n} p_j^0\right)^2 + i\varepsilon} \cdot 2\sum_{k=1}^{n} p_k^0$$

$$\langle \Psi_0 | T\varphi(x)\varphi(0) | \Psi_0 \rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x} \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 2E_i} \cdot (2\pi)^3 \cdot \delta^3\!\left(\vec{k} - \sum_{j=1}^{n} \vec{p}_j\right) |\langle \Psi_0 | \varphi(0) | n \rangle|^2 \frac{i}{(k^0)^2 - \left(\sum_{k=1}^{n} p_k^0\right)^2 + i\varepsilon}\, 2\sum_{l=1}^{n} p_l^0$$

$\Rightarrow$ the propagator in momentum space is

$$D_F(p) = \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3\, 2E_i}\, (2\pi)^3 \delta^3\!\left(\vec{p} - \sum_{j=1}^{n} \vec{p}_j\right) |\langle \Psi_0 | \varphi(0) | n \rangle|^2 \cdot \frac{i}{(p^0)^2 - \left(\sum_{k=1}^{n} p_k^0\right)^2 + i\varepsilon}\, 2\sum_{l=0}^{n} p_l^0$$

Multiplying this by $\int dM^2\, \delta\!\left(M^2 - \left(\sum_{i=1}^{n} p_i\right)^2\right) = 1$ we get (note that $M^2 = \left(\sum_{i=1}^{n} p_i\right)^2 = \left(\sum_{i=1}^{n} p_i^0\right)^2 - \underbrace{\left(\sum_{i=1}^{n} \vec{p}_i\right)^2}_{=\vec{p}^2}$):

$$\boxed{D_F(p) = \int dM^2\, \rho(M^2)\, \frac{i}{p^2 - M^2 + i\varepsilon}}$$

where we have defined the spectral density function $\rho(M^2)$ by

$$\rho(M^2) = \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2E_i}\, \delta\Big(M^2 - \big(\sum_{i=1}^{n} p_i\big)^2\Big) \cdot \Big(2 \cdot \sum_{\ell=0}^{n} p_\ell^0\Big) \cdot \delta^3\big(\vec{p} - \sum_{k=1}^{n} \vec{p}_k\big)\, |\langle \Psi_0 | \varphi(0) | n \rangle|^2 \cdot (2\pi)^3.$$

Note that $\rho(M^2)$ is Lorentz-invariant $\Rightarrow$ we can always calculate it in a frame with $\vec{p} = 0$.

$$\Rightarrow \rho(M^2) = \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2E_i}\, \delta\Big(M^2 - \big(\sum_{i=1}^{n} p_i\big)^2\Big) \cdot \Big(2 \sum_{\ell=0}^{n} p_\ell^0\Big)\, \delta^3\big(\sum_{k=1}^{n} \vec{p}_k\big)\, |\langle \Psi_0 | \varphi(0) | n \rangle|^2 \cdot (2\pi)^3.$$

It is important that $\rho(M^2) \propto \delta\Big(M^2 - \big(\sum_{i=1}^{n} p_i\big)^2\Big)$.

Hence for $n=1$ get $\rho(M^2) \propto \delta(M^2 - \underbrace{p_1^2}_{m^2_{phys}}) = \delta(M^2 - m^2)$

$\Rightarrow$ there is a δ-function peak at $M^2 = m^2$.

$$\rho(M^2) = \delta(M^2 - m^2)\, |\langle \Psi_0 | \varphi(0) | 1 \rangle|^2 \equiv Z\, \delta(M^2 - m^2)$$

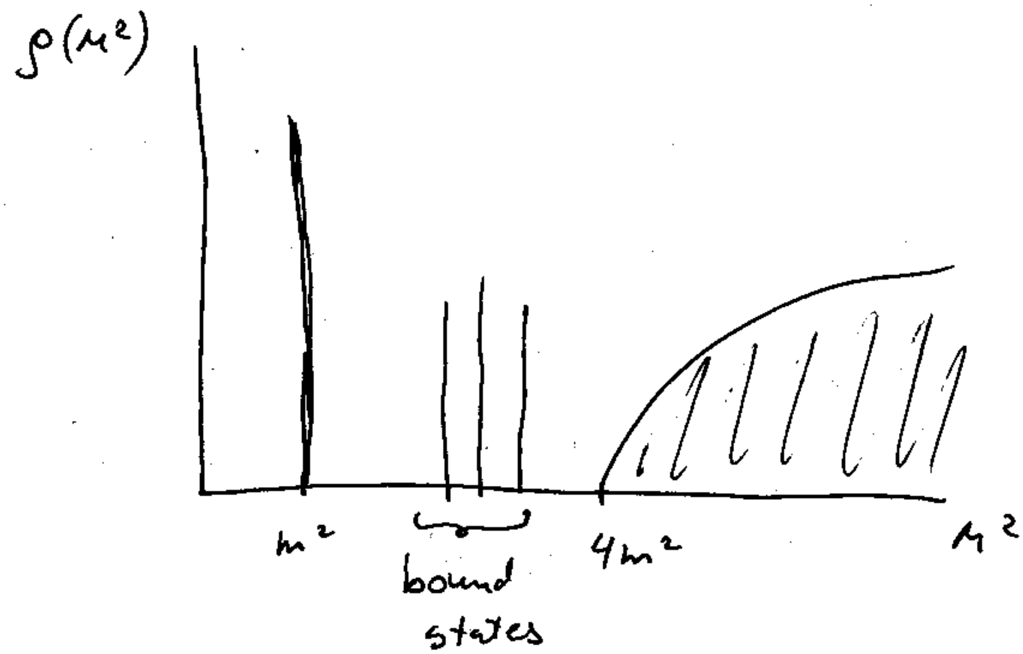
$\uparrow$ field-strength renormalization.

For $n=2$ get $\rho(M^2) \propto \delta(M^2 - (p_1+p_2)^2) =$

$= \delta(M^2 - s)$ where $s \geq 4m^2$ (2-particle threshold)

Note that the energy of the particles is not fixed $\Rightarrow$ s varies from $4m^2$ to ∞.

$\Rightarrow \rho(M^2)$ has a peak at $M^2 = m^2$ & is non-zero for $M^2 \geq 4m^2$. Typically it looks like



In an interacting theory there may also be bound states.

$$\Rightarrow \rho(M^2) = Z\,\delta(M^2 - m^2) + \Theta(M^2 - 4m^2)\cdot(\text{stuff})$$

$$\Rightarrow \boxed{D_F(p) = \underbrace{Z\,\frac{i}{p^2 - m^2 + i\varepsilon}}_{\text{dressed propagator}} + \underbrace{\int_{4m^2}^{\infty} dM^2\,(\text{stuff})\cdot\frac{i}{p^2 - M^2 + i\varepsilon}}_{\text{multi-particle contribution}}}$$